Chapter VI. Mappings of the Inversive Plane

Note. In this chapter we define the "inversive plane" (or the "extended complex plane") and consider one to one and onto mappings (i.e., bijections) of the inversive plane to itself. Similar to Chapter V, we treat the inversive plane as an extension of the complex plane and take advantage of the algebra of complex numbers.

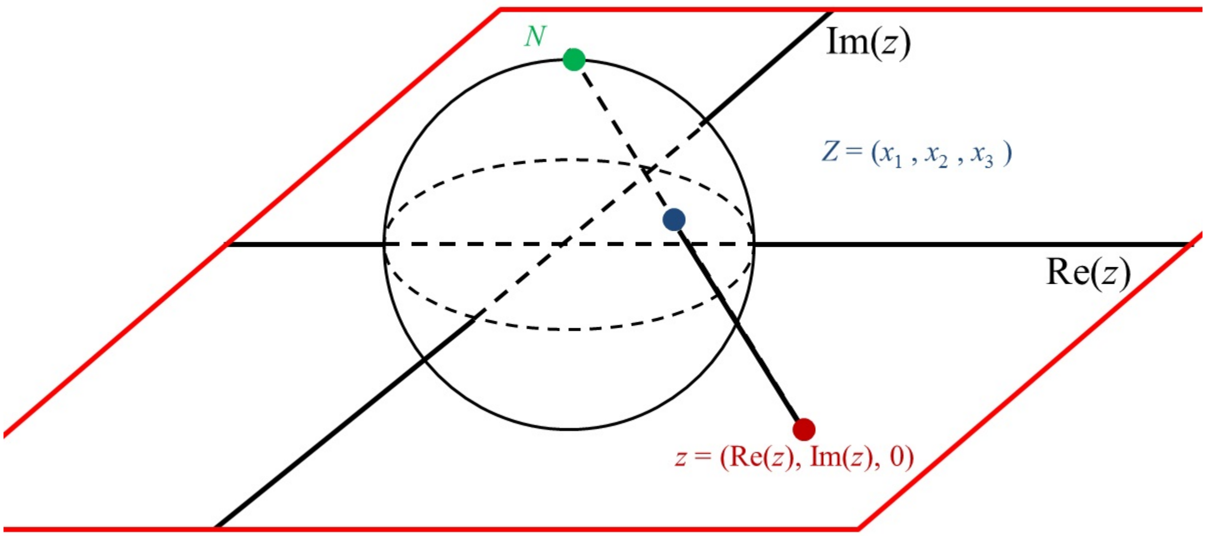
Note. Below, we will add the symbol "∞" to the field of complex numbers $\mathbb{C}$. This is just a symbol and we must avoid arithmetic with this symbol! Pedoe makes some definitions that are extremely undesirable and we shall work around them. Pedoe claims that:

- for all complex numbers c we have $c \pm \infty = \infty$ and $c/\infty = 0$, and
- for non-zero complex c we have $c \cdot \infty = \infty$ and $c/0 = \infty$.

Even Pedoe leaves $\infty - \infty$, ∞/∞, $0 \cdot \infty$, and $0/0$ undefined. **This is not the way to handle this!!!** Pedoe is doing this to give a streamlined way of dealing with the transformations defined in Section 52. Möbius Transformations when applied to ∞ and complex numbers which produce division by 0. We will deal with this in these notes in a rigorous way (though it will take a bit longer).

Definition. The Gauss plane, with the point ∞ adjoined to it, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, is the *inversive plane* (or the *extended complex plane*).

Note. We now introduce a bijection from the sphere in $\mathbb{R}^3$ to the inversive plane $\mathbb{C}_\infty$. The points on the sphere, except for the uppermost point (labeled N in the following figure), are projected onto the plane with a line through N and the point on the sphere. The line intersects the plane at a single point and this point is the projection of the point on the sphere. Finally, the point N on the sphere is mapped to ∞. This mapping is called *stereographic projection* or the *Riemann mapping*.



See my online notes for Complex Analysis 1 (MATH 5510) on Section I.6. The Extended Plane and Its Spherical Representation for more details, including the introduction of a measure of distance between points of the inversive plane, $\mathbb{C}_\infty$. The measure of distance d (called a *metric*) is:

$$d(z, z') = \begin{cases} \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}} & \text{if } z, z' \in \mathbb{C} \\ \frac{2}{\sqrt{1+|z|^2}} & \text{if } z' = \infty. \end{cases}$$

Notice that the maximum distance between any two points of $\mathbb{C}_\infty$ is 2. With the metric, we can talk about open sets, closed sets, compact sets, continuity of functions, and limits (including limits that "converge to ∞"). In my online

Complex Analysis notes Supplement. The Extended Complex Plane (a supplement to Section II.4. Compactness) it is shown that $\mathbb{C}_\infty$ is a compact space under metric d. This space is the "one point compactification" of the Gauss plane $\mathbb{C}$; for details on this idea, see my online notes for Introduction to Topology (MATH 4357/5357) on Section 29. Local Compactness.

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