Chapter 1. The Elements of Perspective

1.1. Introduction

Note. In this section we motivate projective geometry by considering some of the work of Renaissance artists. We give a super-brief history of perspective in art history (a topic well-outside of your instructors areas of expertise!).

Note 1.1.A. The painters of ancient Greece and Rome were somewhat successful with producing paintings that looked three-dimensional.

A first century fresco from a villa in Boscoreale near Pompeii, from the Wikipedia webpage on Graphical Perspective (accessed 10/7/2023)

Medieval European artists produced works which were rather flat in appearance. Spatial relations were ignored (notice in the image below that similar objects are the same size whether near or far, and the white animals all have the same unnatural pose).

A 6th century Mosaic in the Basilica of Sant'Apollinare in Ravenna, Italy from the smarthistory The Center for Public Art History website (accessed 10/7/2023)

Note 1.1.B. With a renewed interest in the literary and artistic works of classical Greece and Rome toward the end of the thirteenth century, artists started to make their works look more natural and realistic. Duccio di Buoninsegna of Tuscany (circa 1255–circa 1318) and Giotto di Bondone (circa 1267–January 8, 1337) of Florence introduced an "intuitive theory of perspective" (as Wylie calls it on page 2). In attempts to convey a three-dimensional image in a two-dimensional painting they tried to give a well-defined plane of the ground, made more distant objects smaller, and introduced converging lines to indicate depth. Works of each of these two artists are given below concerning Christ entering Jerusalem. In Duccio's version, you can see the people in the background are smaller than the ones in the foreground (the small people in the foreground are presumably children) and that the lines of the buildings are to indicate depth. You can also see that there is no clear "vanishing point" for the lines determined by the buildings. In Giotto's version, the image is a bit more flat with almost all of the people drawn in profile (compare this to Duccio's presentation of the faces and heads). The structure on

the right gives an impression of depth, though it is not large enough in the painting to look for converging lines.

Duccio, Christ Entering Jerusalem, 1308–11, from the ArtHistoryBuff.art Giotto and Duccio: Christ entering Jerusalem website (accessed 10/7/2023)

Giotto, Christ entering Jerusalem, 1305–6, from the ArtHistoryBuff.art Giotto and Duccio: Christ entering Jerusalem website (accessed 10/7/2023)

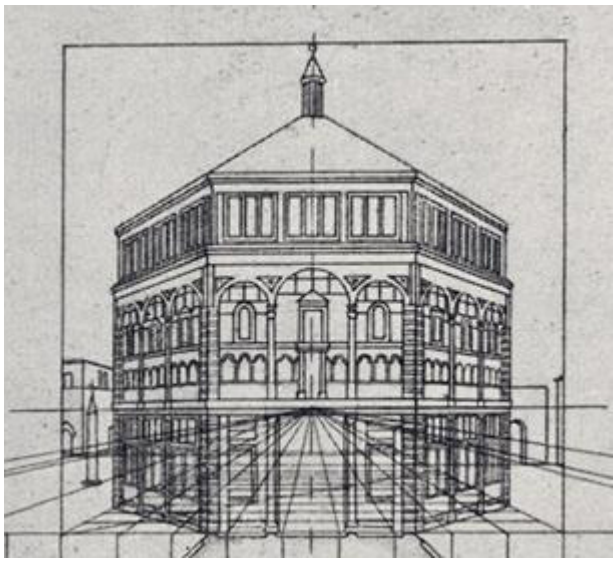
Ambrogio Lorenzetti (circa 1290–June 9, 1348) was an Italian painter who took the intuitive theory of perspective to its culmination. An example of his work is given below in the image of the Presentation at the Temple. Notice the convergence of the lines along the ornamental ceiling. A similar pattern of convergent lines is reflected in the tiles of the floor.

Presentation at the Temple, 1342, from The Uffizi Museum in Florence, Italy.

This image is from The Uffizi Museum webpage (accessed 10/8/2023).

Note 1.1.C. More sophisticated representations of three dimensions on a surface had to wait for the development of a mathematical theory of perspective. Filippo Brunelleschi (1377–April 15, 1446) was best-known as an architect. He designed a

hospital, part of the Basilica of San Lorenzo in Florence, Italy (Michelangelo also worked on designing parts of this Basilica), the Basilica of Santo Spirito in Florence, and the Pazzi Chapel in Florence. But his best-known accomplishment is his design of the dome of the Santa Maria del Fiore in Florence. His architectural and engineering knowledge is indicative of the high level of knowledge of mathematics and geometry that he must have possessed.

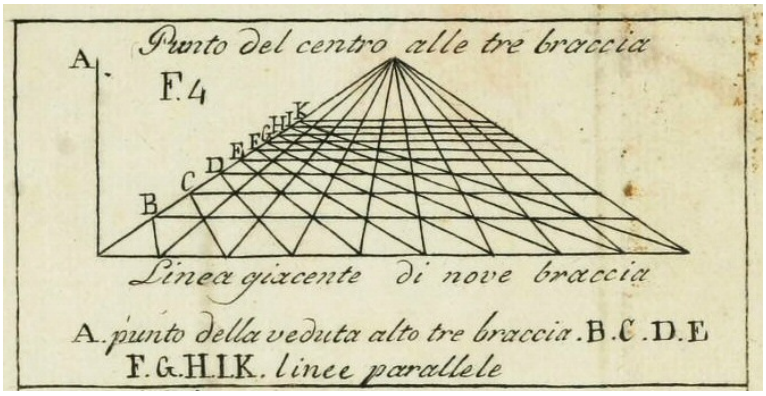


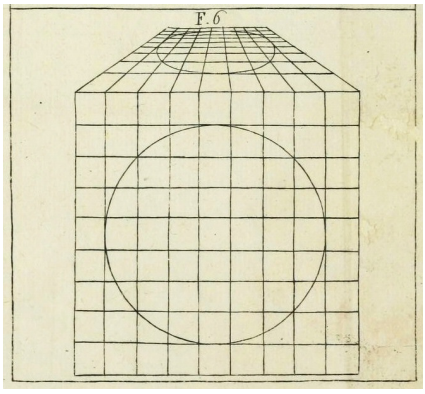
A portrait of Bruenelleshi (left) from the second half of the 15th century from the Wikipedia webpage on Brunelleschi (accessed 10/8/2023). A drawing Brunelleschi made (right) in preparation for a now-lost painting of the Baptistery of Saint John in Florence, Italy. This image is from the Gaining Perspective webpage of Nelson-Atkins.org (accessed 10/8/2023).

(The source for this information on Brunelleschi is largely from the two webpages from which these images come.) We see in Brunelleschi's line drawing above that the lines on the ground all converge to a single point on the horizon (as would any lines in space parallel to those represented by his lines on the ground). This technique of representing such a line is called *linear perspective*. Brunelleschi per-

formed an experiment with mirrors in which he sketched the Baptistery as it would appear in a photograph, leading to the drawing above. After his discovery of linear perspective, other artists quickly adopted his technique.

Note 1.1.D. Leon Battista Alberti (February 14, 1404–April 25, 1472) was a polymath of the 15th century. He was a Renaissance humanist author who also studied architecture, poetry, languages, philosophy, and art. He wrote *De pictura* ("On Painting") in Latin in 1435, but it was not published until 1450. In this work (which he dedicated to Brunelleschi), he writes: "To make clear my exposition in writing this brief commentary on painting, I will take first from the mathematicians those things with which my subject is concerned." This quote is from page 43 of *Leone Battista Alberti, On Painting*, edited by John Richard Spencer (Yale University Press, 1956) (according to the Wikipedia webpage on Lean Battista Alberti; accessed 10/9/2023). In the images below from various editions of *De pictura*, we see on the left that Alberti addresses the vanishing point and on the right that he represents a circle viewed in perspective as a conic section.





Images from the Wikipedia webpage on *De pictura* (accessed 10/9/2023).

These images are from the Wikipedia webpage on *De pictura* (accessed 10/9/2023),

on which part of this notes is based. Compare the image on the left with Figures 1.4 and 1.5 in Section 1.2 Elements of Perspective. Alberit's *De pictura* would influence painters Donatello, Ghiberti, Botticelli, and Ghirlandaio.

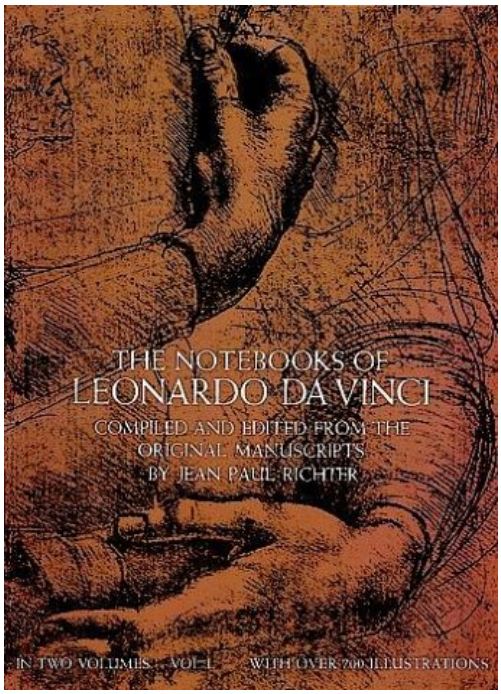
Note 1.1.E. Piero della Francesca (or just "Piero"; circa 1415–October 12, 1492) was an Italian painter, known to his colleagues also as a mathematician and geometer. Three of Piero's mathematical works have survived to our time: *Trattato d'Abaco* ("Abacus Treatise"), *De quinque corporibus regularibus* ("On the Five Regular Solids"), and *De Prospectiva pingendi* ("On Perspective in Painting"). The first is an "abbacus book." These were books meant to educate people on the use of the Hindu-Arabic numerals, place values, and calculating, often by giving numerous examples. Such books flourished in the early part of the second millennium and there may have been as many as a thousand by the fifteenth century. For more on these works, see my online notes for History of Mathematics (MATH 3040) on Supplement. Leonardo of Pisa (Fibonacci) and the *Liber abbaci* (Finonacci's *Liber abbaci* is an early example of an abbacus book). The second book involves the five "Platonic solids" introduced in Book XIII of Euclid's *Elements*; for more on this, see my History of Mathematics notes on Section 5.4. Content of the "Elements" (see Note 5.4.Q). Piero wrote in Latin and this work was translated into Italian by Luca Pacioli (1445–1517) in 1509 as the third volume of Pacioli's *Divine Proportions*, including illustrations by Leonardo da Vinci (April 15, 1452–May 2, 1519); see Note 8.5.F of Section 8.5. The Fifteenth Century in my History of Math notes. Piero gave his mathematical theory of perspective in the third work, "On Perspective in Painting." The image below is Piero's *Brera Modonna* or *Montefel-*

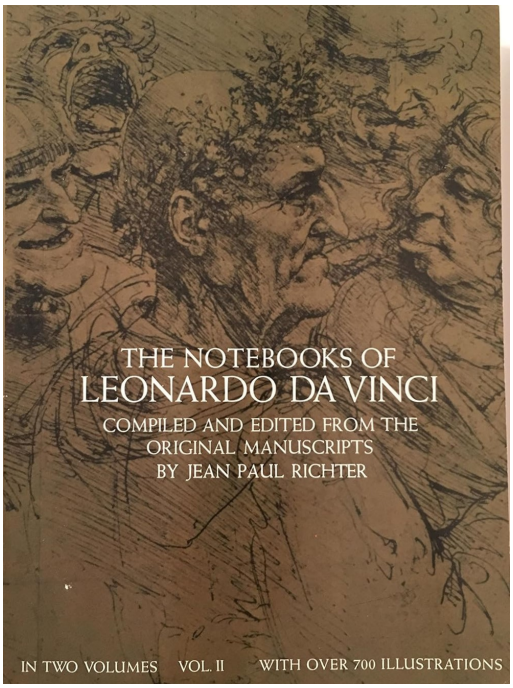
tro Altarpiece that was painted between 1472 and 1474, and today is located in the painting gallery Pinacoteca di Brere in Milan, Italy. The ornamental ceiling and crown molding (as well as the floor, to a lesser degree) shows the use of his theory of perspective and the use of the vanishing point which can be traced to Mary's face by following the crown molding. Piero did additional mathematical work in the 1450s by copying and illustrating several works of Archimedes.

This image is from the Wikipedia webpage on Piero della Francesca (accessed 10/10/2023), which is also the source for this note.

Note 1.1.F. Leonardo da Vinci (April 15, 1452–May 2, 1519) kept an extensive diary containing his thoughts and drawings on human flight, weapons of war, archi-

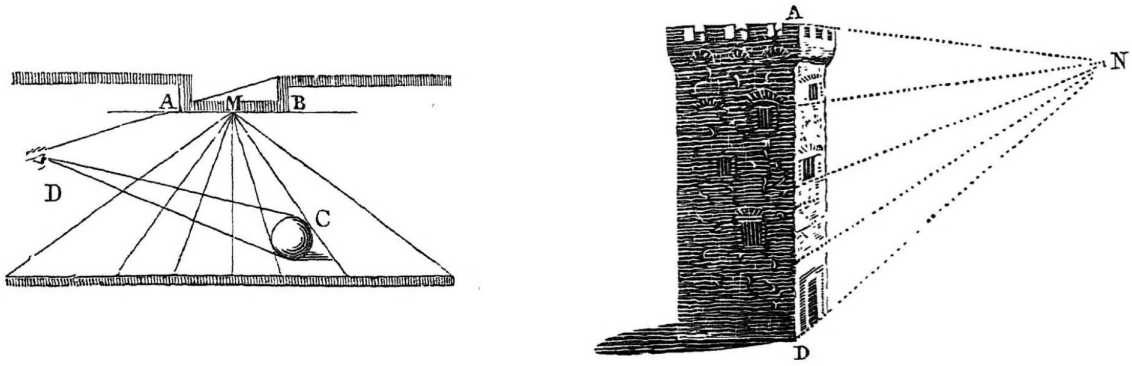
tecture, nature (botany and geology), and, especially, on human anatomy. When Leonardo died, he left his notebooks to his heir Francesco Melzi to be published.





Lenoardo's *Notebooks* are still in print, as seen here in an old Dover paperback version (images from Amazon.com).

Melzi gathered together the papers for *A Treatise on Painting* from 18 of Leonardo's "books" (two-thirds of which have since gone missing). The papers were passed down to Melzi's son. In 1651, an abridges version was published in French version and in Italian as *Trattato della pittura* ("A Treatise on Painting"). Melzi's version was rediscovered in the Vatican Library and published in its modern form in 1817. Much of the book covers human anatomy (and poses), but it also discusses light and shadows, colors, and composition. You can view the book online on the archive.org website (accessed 10/10/2023); this is the source of the following images.



"How the Painter is to place himself in regard to the Light and his Model" (left) and "Of Buildings seen in Thick Air" (right)

The source for this not is the Wikipedia page on *A Treatise on Painting* (accessed 10/10/2023).

Note 1.1.G. Albrecht Dürer (May 21, 1471–April 6, 1528) was a German painter, who also wrote a mathematical treatise on geometry which included a mathematical theory of perspective. This work was *Four Books on Measurement*. The first book is on the geometry of lines, and includes constructions of helices, conchoids, and epicycloids. The second book is on two-dimensional geometry and the construction of regular polygons. The third book applies geometry to architecture, engineering, and typography (i.e., the geometric construction of letters). The fourth book is on three-dimensional geometry and the construction of polyhedra, including the five Platonic solids and some of Archimedes' semi-regular solids. You can view a few pages of *Four Books on Measurement* on the Utah State University Digital Exhibit on Dürer: His Life and Works: Instruction on Measurement (accessed 10/10/2023). He also wrote *Four Books on Human Proportion*, in which he considers human anatomy so that his paintings will look more realistic. This note is based on the Wikipedia webpage on Albrecht Dürer (accessed 10/10/2023); the following self

portrait of Albrecht Dürer at age 26 is also from this source.

Note 1.1.H. Italian painter and architect Raffaello Sanzio da Urbino (1438–April 6, 1520), or "Raphael," is not mentioned in the book, but his work "The School of Athens" must be mentioned.

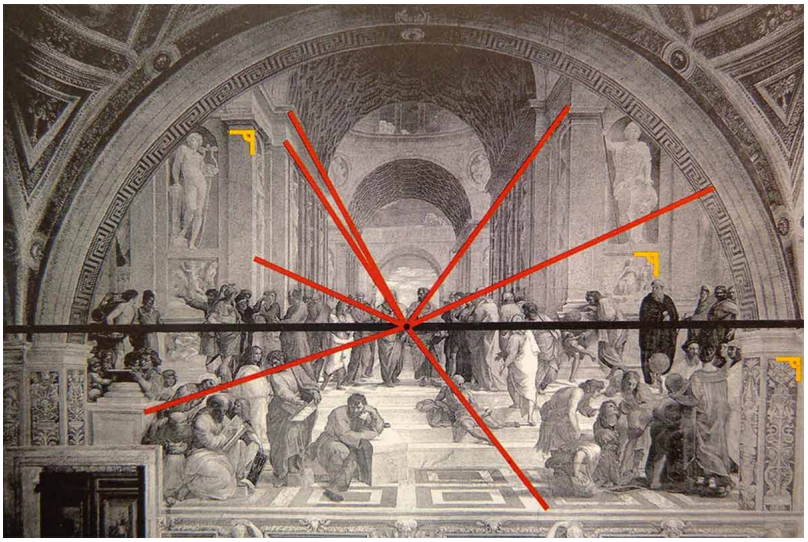


Image from the Widewalls webpage on Perspective in Art—Conjuring the Space (accessed 10/10/2023)

In the image above, we follow the lines determined by the crown molding and the pattern in the floor, we find the vanishing point and see that coincides in the painting with the main characters, Plato and Aristotle. This painting oozes depth! It includes some very famous mathematicians of the classical world. Pythagoras is in the lower left writing in the book, with a young Archimedes just to the right of him. Euclid is in the lower right, bent over drawing a circle. Ptolemy is to the right of Euclid with his back to us. Socrates is also present; he is five or six people to the left of Plato in a brown robe. A larger, clearer image is below.

Raphael's "The School of Athens" from RaphaelPaintings.org (accessed 10/10/2023), painted 1509–1511.

Note 1.1.I. Wylie states on page 2 (with some modernization of the pronouns):

> "Reduced to its simplest terms, the theory of perspective regards the artist's canvas as a transparent screen through which [they look] from a fixed vantage point at the scene [they are] painting. Light rays coming from each point of the scene are imagined to enter [their] eye, and the totality of these lines is called a **projection**. The point where any line in the projection pierces the viewing screen is the image of the corresponding point in the scene being painted. The totality of all these image points, which, of course, becomes the painting itself, is called a **section** of the projection and conveys to the eye the impression as the scene itself."

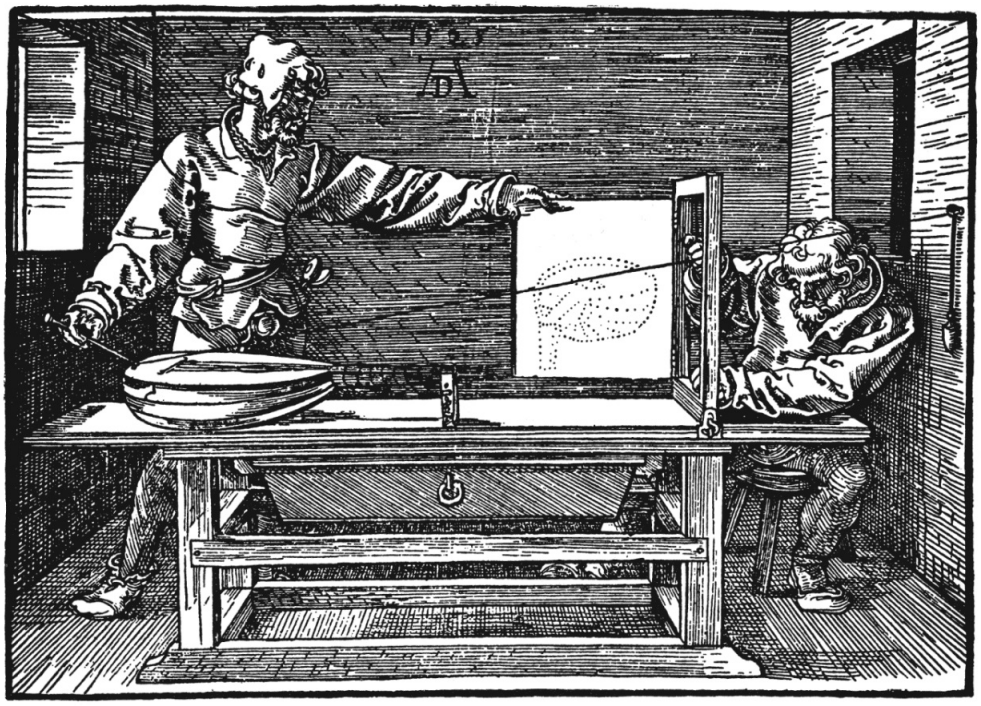


Fig. 1.2 Woodcut from Dürer's "A Treatise on Measurement."

A figure from Dürer's *A Treatise on Measurement* is given above illustrating, apparently, a device based on a string, which represents a line of sight, with one

end (on the right) anchored at a point (where the eye would be) and the other moved around an item (the object on the table, left), and a frame that contains the canvas of the painting. The person on the left moves the end of the string around the object, while the person on the right locates a point in the plane of the canvas, which is transferred as a dot on the canvas itself by rotating it back into the frame. We see in the figure several points on the canvas which outline the object (a musical instrument like a lute, maybe?). In the terminology introduced above, the string in its various positions represents the "projection," and the "section" is the canvas with the dots on it (which ultimately leads to the painting). With this as inspiration, we study perspective in the rest of Chapter 1 much in the spirit of the artists mentioned in this section.

Revised: 10/23/2023