1.2. The Elements of Perspective

Note. In this section, we give several definitions, illustrations for the definitions, and two theorems (with proofs to be given as exercises). As mentioned at the end of the previous section, we are approaching the study of perspective in the spirit of the artists of the early renaissance.

Note 1.2.A. Consider the setup given in Figure 1.3 below. We introduce a fixed point C called the *viewing point*, a vertical fixed plane ρ called the *picture plane* or *image plane*, and (for simplicity) a two-dimensional scene lying in plane σ (called the *object plane*) which is horizontal and perpendicular to the picture plane. The line joining the viewing point to any point P in the object plane is the *line of sight* or *projecting line* of P. The point P' in which the line of sight from P intersects the picture plane is the *image* of P. Notice that point P in the object plane and its image P' in the picture plane are collinear with the viewing point C. Such a mapping is called a *perspective transformation*. (In this artistic model, the part of the object plane being viewed is presumably on the side of the picture plane opposite that of the viewing point. In a more mathematical approach, we need not make this assumption.) The point of intersection of the picture plane ρ and the line through the viewing point which is perpendicular to to ρ is the *principal vanishing point*, V. The line v which is the intersection of the picture plane and the plane through C which is parallel to the object plane is the *vanishing line* or *horizon line*.

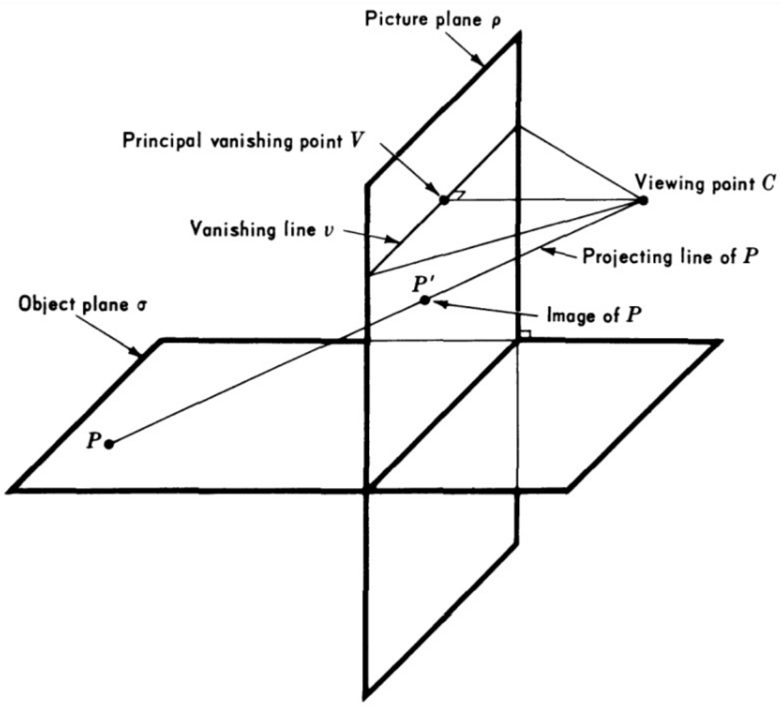


Fig. 1.3 The elements of a perspective transformation.

Comparing this to Albrecht Dürer's woodcut in Figure 1.2 of Section 1.1. Introduction, we see that the picture plane here corresponds to the frame into which the canvas is inserted there, the object plane here corresponds to the object on the table there (strictly speaking, though, the object is not two-dimensional and does not lie flat on the table), the line of sight here corresponds to the string there, and point P' here corresponds to the point in the frame which the person on the right is locating in Dürer's woodcut.

Note 1.2.B. Consider the case where the object plane contains a railroad track which is perpendicular to the picture plane, as in Figure 1.4. Let the rail be given by line l_1 in the object plane σ. The image of any point on the rail corresponding to line l_1 is found by joining the point to the viewing point C with a line (the line of sight) and finding the intersection of this line with picture plane ρ. Now in three dimensions, a line and a point not on the line determine a unique plane containing the line and point. So there is a unique plane, π_1 (not in Figure 1.4), containing line l_1 and point C. Two distinct non-parallel planes in three dimensions intersect in a line, so the plane π_1 containing l_1 and C intersects the picture plane in a line, and the image of l_1 is the line segment l_1' as given in Figure 1.4. Line l_1 and the line through points C and V are parallel (both are perpendicular to the picture plane), and hence lie in some plane.

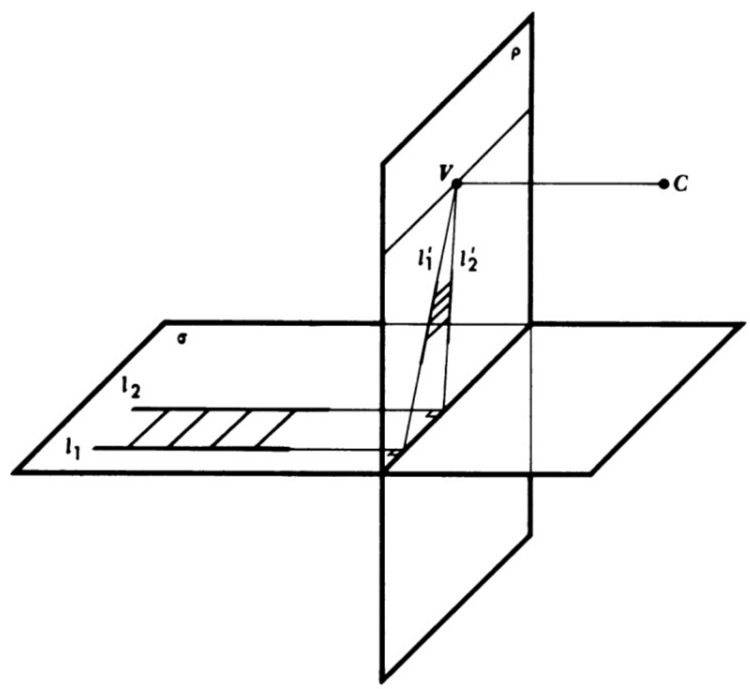


Fig. 1.4 The significance of the principal vanishing point in a perspective transformation.

This plane contains the principal vanishing point V, so the intersection of this plane and the picture plane both contain point V. Wiley says (on page 6) that "the image, l_1', of the line l_1 passes through V." If we *construct* l_1' by projecting all points of l_1 onto the picture plane, then this is not true. There is no point on l_1 corresponding to point V. However, V is a *limit* of points on l_1'. In fact, if the line l_1 is extending without bound in both directions, then the projection process will give all points on the line containing l'_1 except for point V (we might say that V correspond to "the point at infinity" under the projection of line l_1). This is addressed in Exercise 1.2.B(a). Of course similar properties hold for the projection l'_2 of l_2. The next theorem summarizes these observations. A rigorous proof of the theorem requires more quantitative equipment (such as a coordinate system and the equations of the relevant lines). This is developed in Exercise 1.2.A, and a proof of the following theorem is to be given in Exercise 1.2.B(b).

Theorem 1.2.1. All lines in the scene which are perpendicular to the picture plane appear on the picture plane as lines which pass through the principal vanishing point in the picture plane.

Note. Theorem 1.2.1 gives the importance of the principal vanishing point. The next argument and the theorem which follows it gives the importance of the vanishing line.

Note 1.2.C. Now consider two parallel lines m_1 and m_2 in the object plane (such as train tracks) that are not perpendicular to the picture plane and not parallel to the vanishing line. See Figure 1.5. The argument given in Note 1.2.B applies here as well to show that the projections of the points on m_i (where $i \in \{1, 2\}$ all lie in the plane π_i determined by C and m_i. So the image of m_i under projection, m'_i, lies on the intersection of π_i and the picture plane ρ. As before, there is a unique line through the viewing point C which is parallel to m_i and this line lies in plane π_i. This line is horizontal, as is the plane determined by the viewing point C and the vanishing line v, so that this line must intersect the vanishing line at some point V_m. Point V_m lies both in plane π_i (determined by C and m_i) and the picture plane ρ, and so it must lie on the line of their intersection. Now the projection of m_i, m_i, also lies on the intersection of π_i and ρ. As in Note 1.2.B, we have that point V_m is a *limit point* of m_i' (though, again, Wylie says on page 7 that "the point V_m... must lie on their intersection, m_i'"; there is again the "point at infinity" problem and Wylie gives some indication of this at the bottom of page 7 and the top of page 8 when he mentions that the projections of such lines "converge" to points on the vanishing line).

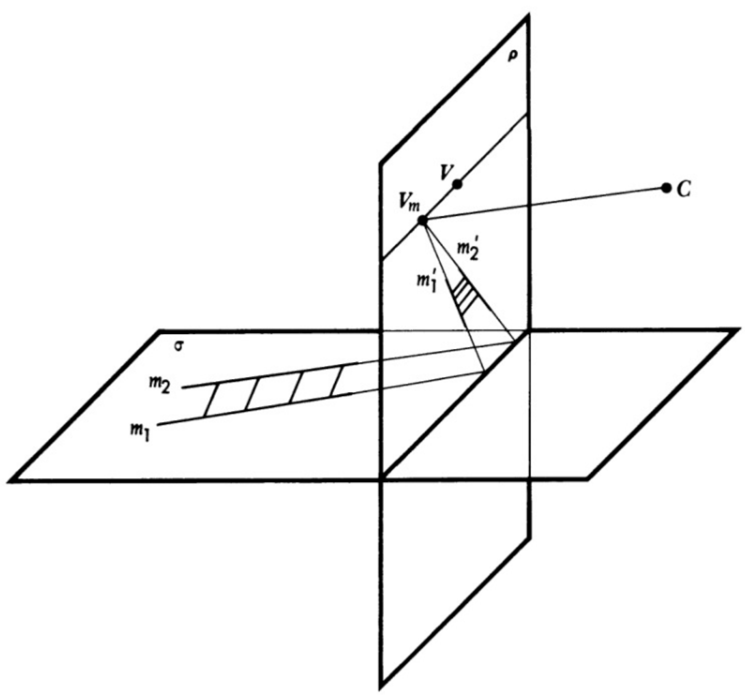
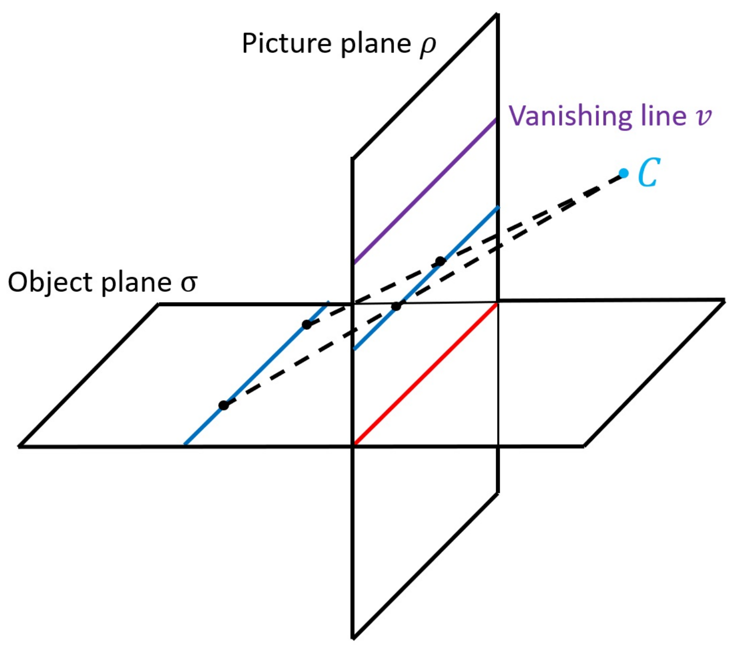


Fig. 1.5 The mapping of a general pair of parallel lines in a perspective transformation.

Note 1.2.D. If a line in the object plane is parallel to the line of intersection of the object plane σ and the picture plane ρ), then the perspective transformation maps such a line to a line parallel to the vanishing line in the picture plane. That is, such a line has an image under perspective transformation has an image that does not intersect the vanishing line. So if we have a family of lines in the object plane parallel to the intersection of σ and ρ, then the perspective transformation transforms them to a family of lines parallel to the vanishing line. See the figure below. These observations are summarized in Theorem 1.2.2. A proof is to be given in Exercise 1.2.B(c).



Theorem 1.2.2. The lines of a general parallel family lying in a plane perpendicular to the picture plane, but not parallel to the vanishing line, appear on the picture plane as lines which pass through a unique point on the vanishing line in the picture plane.

Note. We next consider an example that illustrates how to quantify (and hence justify) some of the claims made above.

Example 1.2.1. Consider the situation given in Figure 1.6. We have introduced a three-dimensional coordinate system into the settings described in Figures 1.4 and 1.5. The viewing point C is taken to be the point $(0,-3,2)$, the picture plane is the plane $y=0$, and the object plane is the plane $z=0$. We want to find several images. **(a)** What is the image on the picture plane of the point $(2,1,0)$? **(b)** What is the image of the family of parallel lines $y=x+k$ in the object plane? **(c)** What is the image of the circle, Γ, whose equation in the object plane is $x^2+(y-2)^2=1$?

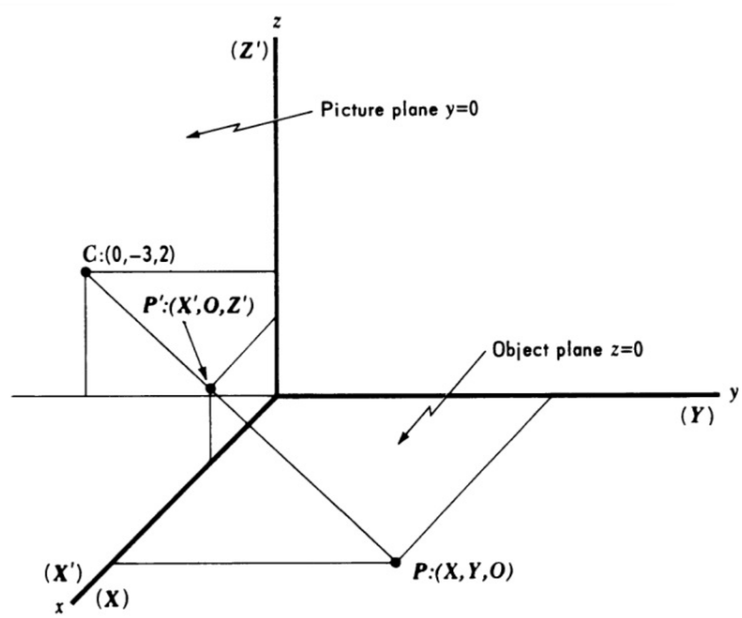


Fig. 1.6 The data for a particular perspective transformation.

Solution. As indicated in Figure 1.6, in addition to coordinates x, y, z, we also introduce auxiliary coordinates X, Y in the xy-plane (i.e., the object plane $z=0$), and we introduce coordinates X', Z' in the xz-plane (i.e., the picture plane $y=0$).

As in Figure 1.6, for point $P : (X, Y, 0)$ in the object plane, we denote the point of projection in the picture plane as $P' : (X', 0, Z')$.

(a) If a line three dimensions contains points (x_0, y_0, z_0) and (x_1, y_1, z_1), then the points (x, y, z) on the line satisfy

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}.$$

You see something similar to this in Calculus 3 (MATH 2110) where a line through point (x_0, y_0, z_0) with direction vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is given parametrically as

$$x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3,\ t \in (-\infty, \infty).$$

See my Calculus 3 online notes on Section 12.5. Lines and Planes in Space. In our setting, the direction vector is $\mathbf{v} = (x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}$, so that if we solve for the parameter t in each of the component equations of the parametric representation, then we get the three equal quantities given above. Taking the viewing point $C : (0, -3, 2)$ as the point (x_0, y_0, z_0) and $P : (X, Y, 0)$ as the point (x_1, y_1, z_1), then we have that the projecting line PC is given by the equations

$$\frac{x - 0}{X - 0} = \frac{y + 3}{Y + 3} = \frac{z - 2}{0 - 2}.$$

Projection point $P' : (X', 0, Z')$ lies on the projecting line, so we set $x = X'$, $y = 0$, and $z = Z'$ in the equations to get

$$\frac{X'}{X} = \frac{3}{Y + 3} \text{ and } \frac{3}{Y + 3} = \frac{Z' - 2}{-2},$$

$$\text{or } X' = \frac{3X}{Y + 3} \text{ and } Z' - 2 = \frac{-6}{Y + 3}, \text{ or } Z' = \frac{2Y}{Y + 3}. \qquad (1)$$

Now with $P : (X, Y, 0) = (2, 1, 0)$, we now have $X' = \dfrac{3(2)}{(1) + 3} = \dfrac{3}{2}$ and $Z' =$

$\dfrac{2(1)}{(1)+3} = \dfrac{1}{2}$. That is, the image of $P : (2, 1, 0)$ under this projection is the point $\boxed{(3/2, 0, 1/2)}$.

(b) Now we consider the family of parallel lines in the object plane of the form $y = x + k$ or, in the auxiliary coordinates of the object plane, $Y = X + k$. First, we rearrange the equations of (1) we have

$$X'(Y + 3) = 3X, \text{ and } Z'(Y + 3) = 2Y \text{ or } Y(Z' - 2) = -3Z' \text{ or } Y = \frac{3Z'}{2 - Z'}.$$

Substituting the last of these equations into the first one gives

$$X'\left(\frac{3Z'}{2 - Z'} + 3\right) = 3X \text{ or } X = X'\frac{3Z' + 6 - 3Z'}{3(2 - Z')} = \frac{6X'}{3(2 - Z')} = \frac{2X'}{2 - Z'}.$$

That is, the coordinates of $P : (X, Y, 0)$ in the object plane can be expressed in terms of the coordinates of its image under projection $P'(X', 0, Z')$ as

$$X = \frac{2X'}{2 - Z'} \text{ and } Y = \frac{3Z'}{2 - Z'}. \tag{2}$$

The line $Y = X + k$ in the object plane then projects into the picture plane as

$$\frac{3Z'}{2 - Z'} = \frac{2X'}{2 - Z'} + k \text{ or } \frac{3Z'}{2 - Z'} = \frac{2X'}{2 - Z'} + \frac{k(2 - Z')}{2 - Z'}.$$

Next, we multiply through by $2 - Z'$ and cancel. However, we see from the previous equation that Z' cannot be 0, so this gives us $3Z' = 2X' + 2k - kZ'$ where $Z' \neq 2$. This simplifies to $(3 + k)Z' = 2X' + 2k$ where $Z' \neq 2$. Notice that both the geometry and the algebra forbid Z' from taking on the value 2. This is because $Z' = 2$ (which implies that $X' = 2$) corresponds to a point on the vanishing line, $(X', 0, Z') = (3, 0, 2)$. However, as observed in Notes 1.2.B and 1.2.C, the points on the vanishing line are *limit points* of the projections of lines, but are not themselves *projections* of points in the object plane. In spite of all this, Wylie reports the solution to this as $\boxed{(3+k)Z' = 2X' + 2k}$ without the constraint on Z'. From the practical point of view, the presence or absence of a single point (or even a line) in a painting has no effect on the appearance of the painting. So, by convention, when describing projections in this setting we will include the points on the vanishing line.

(c) Similar to (b), to project the circle Γ in the object plane given by $x^2+(y-2)^2 = 1$ (or $X^2 + (Y-2)^2 = 1$ in the auxiliary coordinates), we again use the equations given in (2). This gives

$$\left(\frac{X'}{2-Z'}\right)^2 + \left(\frac{3Z'}{2-Z'} - 2\right)^2 = 1 \text{ or } (2X')^2 + (3Z' - 2(2-Z'))^2 = (2-Z')^2$$

or $4(X')^2+(5Z'-4)^2 = 4-4Z'+(Z')^2$ or $4(X')^2+25(Z')^2-40Z'+16 = 4-4Z'+(Z')^2$

$$\text{or } 4(X')^2 + 24(Z')^2 - 36Z' + 12 = 0 \text{ or } (X')^2 + 6(Z')^2 - 9Z' + 3 = 0.$$

Notice that we next have $(X')^2 + 6((Z')^2 - (3/2)Z') + 3 = 0$ or $(X')^2 + 6((Z')^2 - (3/2)Z' + (9/16)) + 3 - 6(9/16) = 0$ or $(X')^2 + 6((Z') - (3/4))^2 + 3 - (27/8) = 0$ or $(X')^2 + 6((Z') - (3/4))^2 = 3/8$ or $\boxed{\frac{(X')^2}{3/8} + \frac{((Z') - (3/4))^2}{3/48} = 1}$. So in the picture plane, the circle has projection in the picture plane which is an ellipse with center $(X', Z') = (0, 3/4)$, semimajor axis of length $\sqrt{3/8}$, and semiminor axis $\sqrt{3/48} = 1/4$. $\square$

Exercise 1.2.1. In Example 1.2.1, does the center of the ellipse which is the image of circle Γ coincide with the image of the center of Γ?

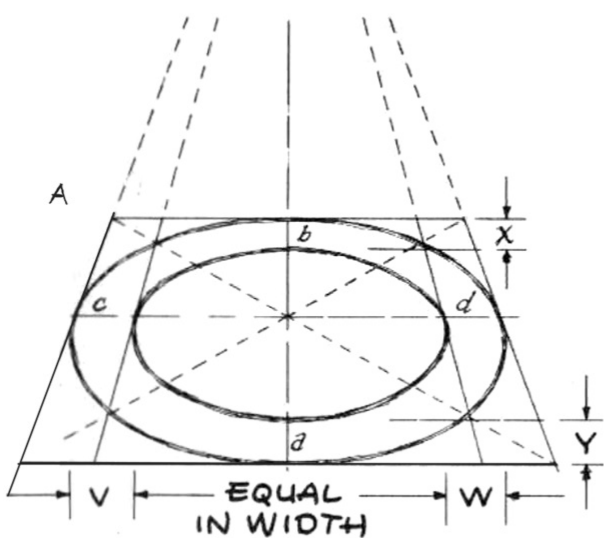
Solution. In auxiliary coordinates, the equation of the circle Γ is $X^2+(Y-2)^2 = 1$, so that the center is $(X, Y) = (0, 2)$. From equation (1) the image of the center of the circle satisfies

$$X' = \frac{3X}{Y+3} = \frac{3(0)}{(2)+3} = 0, \quad Z' = \frac{2Y}{Y+3} = \frac{2(2)}{(2)+3} = \frac{4}{5}.$$

That is, the image of the center of circle Γ is $(X', Y') = (0, 4/5)$. The equation of the image of Γ is the ellipse $\dfrac{(X')^2}{3/8} + \dfrac{((Z') - (3/4))^2}{3/48} = 1$ (as shown in Example 1.2.1), which has center $(X', Z') = (0, 3/4)$. So $\boxed{\text{NO}}$, the centers do not coincide.

□

Note 1.2.E. In "Answers to Odd-Numbered Exercises" (see page 523), the center of the ellipse is given as $(0, 3/2)$. This cannot be the case, as we now argue based on geometry (but also because of the algebraic argument given above). A circle viewed in perspective is an ellipse, as you likely know. When viewing a circle in perspective (as opposed to viewing it "face on"), the radius on the back half of the circle is farther from us than the radius on the nearer half so that the back half appears shortened as compared to the front half. Consider the following image from the drawinghowtodraw.com website on "Drawing Circles in Perspective".

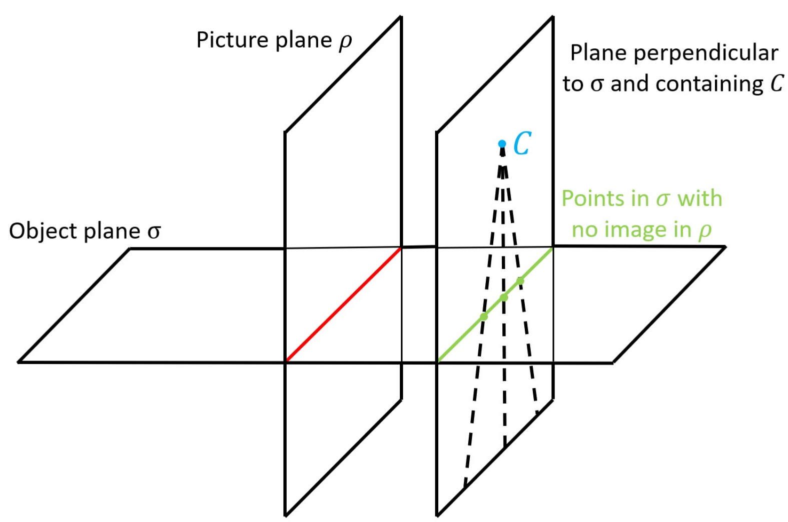


The dotted lines cross at the center of the two concentric circles (in the object plane; here the plane of the notes is the picture plane). However, we see in the picture plane that this point lies above the center of the two ellipses. You can also see that the distance from the top of one of the circles (say the inner one) to the center appears shorter than the distance from the bottom of that same circle to the center. Notice that this means for two-dimensional coordinates in the plane of the notes, the second coordinate of the image of the center of the circle in is *greater than* the second coordinate of the ellipse. That is, in the picture plane the image of the center of the circle is *above* the center of the ellipse. This is consistent with Exercise 1.2.1 in which the image of the circle Γ has picture plane coordinates $(0, 4/5)$ and the center of the ellipse (i.e., the image of circle Γ) has coordinates $(0, 3/4)$; $4/5 = 0.8 > 0.75 = 3/4$). However, the answer in the back of the book where the center of the ellipse is given as $(0, 3/2)$ is inconsistent with this geometric argument (since $4/5 = 0.8 < 1.5 = 3/2$). This idea concerning the location of the center of a circle viewed in perspective is addressed in my publication: R. Gardner and R. Davidson, "Mathematical Lens: The Three Stooges Meet the Conic Sections," *Mathematics Teacher*, **105**(6) (February 2012), 414–418. This can be viewed on my online publications website. Notice Figure 2 in this publication. Also, in Example 1.2.1 we have the major axis of the ellipse along the X' axis and the minor axis along the Z' axis, as expected.

Note. We work one more exercise. We rely on informal geometric arguments in three-dimensions, since we have do not yet have much quantitative equipment.

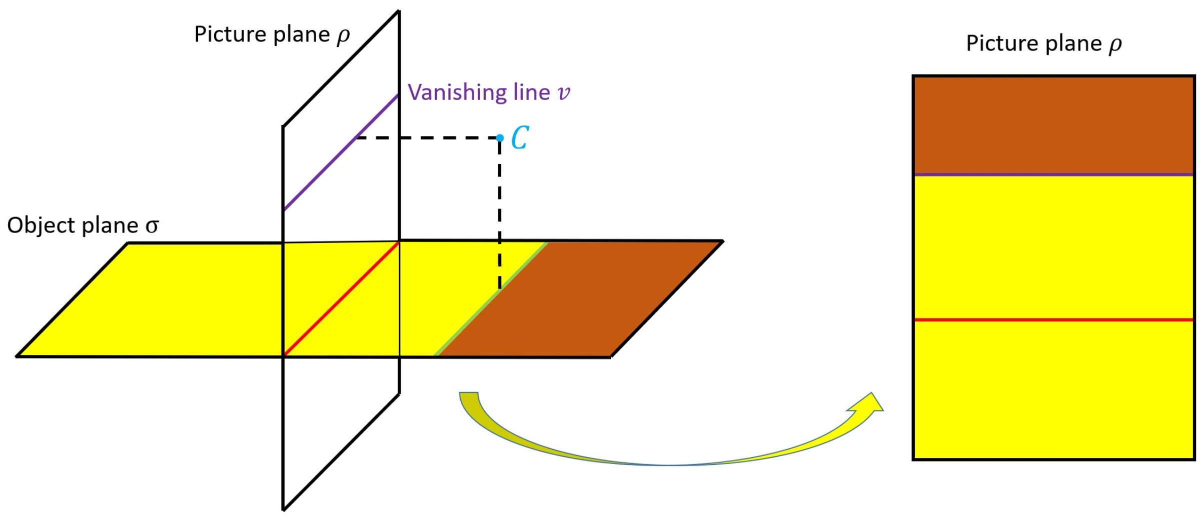
Exercise 1.2.8. **(a)** In a perspective transformation, is every point in the picture plane the image of some point in the object plane? **(b)** Does every point in the object plane have an image in the picture plane? *Hint:* Remember that the object plane extends on both sides of the picture plane.

Solution. **(a)** $\boxed{\text{No.}}$ We addressed this in Note 1.2.B. The points on the vanishing line are *limit points* of lines in the object plane which are not parallel to the line of intersection of the picture plane and the object plane, but they are not *images* of points in the object plane, because the projecting line through points on the vanishing line are parallel to the object plane and so do not intersect it.
(b) $\boxed{\text{No.}}$ A point in the object plane has an image in the picture plane *if* the projecting line passes through the point in the object plane and the viewing point, and this line intersects the picture plane.



So if a projecting line is parallel to the picture plane, then the point at which it intersects the object plane (which it must do since the object plane is perpendicular to the picture plane) has no image in the picture plane. This is the case for all points in the object plane and in the plane which is perpendicular to the object plane and contains the viewing point C. The line of intersection of these two perpendicular planes will be called the *vanishing line* of another transformation which we will see in the next section (the transformation there is called the "plane perspective").

Note 1.2.F. To further illustrate the projective transformation, we color code the object plane and see how various lines and regions are mapped into the picture plane. Notice that the green line in the object plane has no image in the picture plane.



Note 1.2.G. In order to give proofs of the claims of this section, we need more quantitative information about the perspective transformation. If we take the viewing point as the arbitrary point $C : (x_c, y_c, z_c)$ in Example 1.2.1, the we find that a point $P : (X, Y, 0)$ in object plane is mapped by the perspective transformation to $P' : (X', 0, Z')$ where we have the relationships:

$$X' = \frac{-y_c X + x_c Y}{Y - y_c} \text{ and } Z' = \frac{z_c Y}{Y - y_c},$$

and

$$X = \frac{-z_c X' + x_c Z'}{Z' - z_c} \text{ and } Y = \frac{y_c Z'}{Z' - z_c}.$$

This is to be shown in Exercise 1.2.A. This result can then be used to prove Theorems 1.2.1 and 1.2.2 (see Exercise 1.2.B). It can also be used to see that the perspective transformation maps conic sections to conic sections (see Exercise 1.2.C(a)). We can treat the first relationships as a transformation from the XY-plane to the $X'Z'$ plane defined as

$$f(X,Y) = \left(\frac{-y_c X + x_c Y}{Y - y_c}, \frac{z_c Y}{Y - y_c} \right) = (X', Z').$$

We can treat the second relationships as a transformation from the $X'Z'$-plane to the XY-plane defined as

$$f^{-1}(X', Z') = \left(\frac{-z_c X' + x_c Z'}{Z' - z_c}, \frac{y_c Z'}{Z' - z_c} \right) = (X, Y).$$

We can easily show that $(f^{-1} \circ f)(X,Y) = (X,Y)$ if $Y \neq y_c$ and $y_c \neq 0 \neq z_c$, and $(f \circ f^{-1})(X', Y') = (X', Y')$ if $Y' \neq y_c$, and $y_c \neq 0 \neq z_c$ (this is to be shown in Exercise 1.2.D). In both cases it is necessary that $y_c \neq 0$ and $z_c \neq 0$. In the event that $y_c = 0$, the viewing point C lies in the picture plane so that the whole object plane projects onto point C; this is a degenerate case and neither projection transformation f nor f^{-1} are one to one so that inverses do not exist. In the event that $z_c = 0$, the viewing point C lies in the object plane and the projection transformation maps the object plane onto the line of intersection of the object plane and the picture plane; this is also a degenerate case and neither projection transformation f nor f^{-1} are one to one so that inverses do not exist. In the first case we need $Y \neq y_c$. In the event that $Y = y_c$ then point $P : (X, Y, Z)$ lies in the vertical plane containing viewing point C, as illustrated in Exercise 1.2.8(b) above, and point P has no image. Such a point is not in the domain of f and so an inverse f^{-1} cannot be defined on the image of such a point P. We need $Y' \neq y_c$ in the second case for a similar reason.

Revised: 10/22/2023