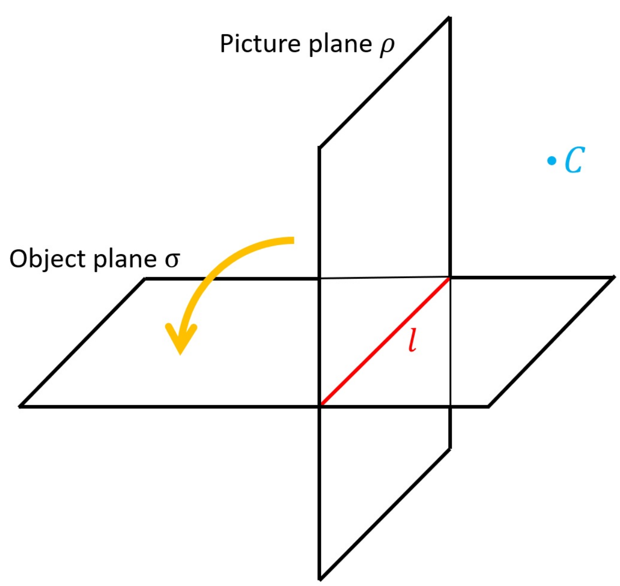
1.3. Plane Perspective

Note. In this section, we define the process of rabattement that rotates the picture plane onto the object plane. We consider the points invariant under projection followed by rabattement and this is used to define the plane perspective transformation. We state and prove two theorems on the plane perspective.

Note 1.3.A. Starting with the configuration of Figure 1.3, we rotate the picture plane about its intersection with the object plane through 90° until its upper half coincides with the half of the object plane on the opposite side of the picture plane from the viewing point C. This process is called *rabattement*. See the figure below. If l is the line of intersection of the object plane and the image plane, then the points of l coincide with their images under projection and are fixed by the rotation.



Next, let π be the plane which passes through the viewing point C and is perpendicular to both the object plane and the picture plane in the original (unrotated) configuration. Let r be the line in this plane which passes through the viewing point C and makes and angle of 45° with the object plane on the opposite side of the picture plane from C as shown in Figure 1.7. Then with point O' as the point of intersection of r and the picture plane ρ, the 90° rotation will place O' on its preimage O in the object plane. In this way, point O and the points on line l are *invariant* in the sense that the image of these points coincides with the points themselves through rabattement. In fact, these are the only such invariant points.

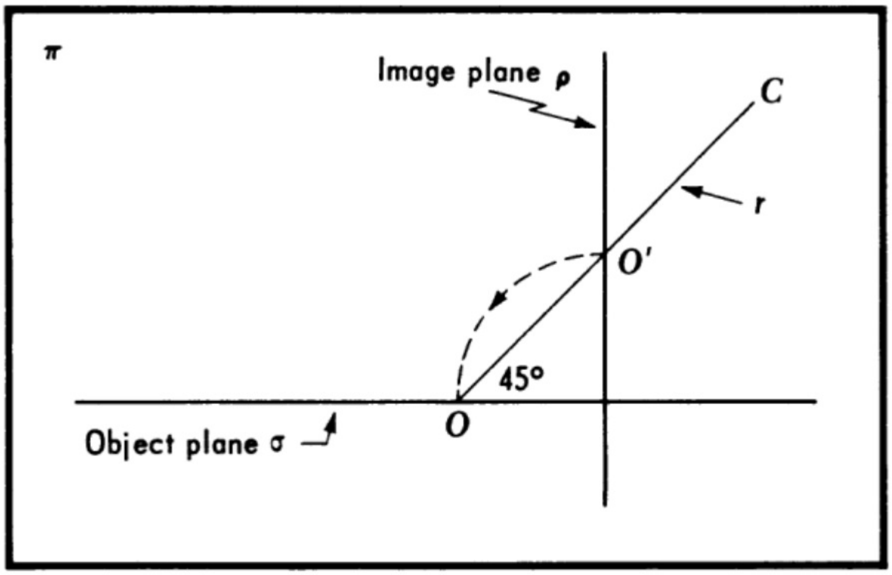


Fig. 1.7 **The process of rabattement.**

Note 1.3.B. For point O above not on the invariant line l, the image of any line in the object plane which passes through O is mapped to itself under projection followed by rabattement (that is, such a line is invariant), which we now argue. Under projection (i.e., the perspective transformation), a line in the object plane is transformed into a line in the picture plane (by Exercise 1.2.B(a)). So the image of a line is uniquely determined by the image of two points on the line. If such a line in the object plane contains point O and is not parallel to line l, then it contains both point O and a point on line l. Since both O and the point on l are invariant under projection followed by rabattement, then the line containing O and not parallel to l is invariant. The line p which passes through O and is parallel to line l is also invariant, because all other lines through O are invariant and the image of p under projection followed by rabattement is some line that contains O. But all lines through O are accounted for, so that the image of p must be p itself. That is, p is also invariant so that all lines through point O are invariant. Notice that it is the lines (i.e., the *sets* of points making up the lines) which are invariant, and not the *points* on those lines which are invariant. As Wiley puts it (on page 13) "each line on O is invariant as a whole but is not point-by-point invariant." We summarize this below in Theorem 1.3.A.

Note 1.3.C. Consider a point $P \neq O$ in the object plane σ. The line OP is invariant under projection followed by rabattement, as described in Note 1.3.B. Therefore the image P' of P under this transformation also lies on line OP. To find the precise location of P', we need to known one more (noninvariant) point G and its image G' under the transformation. We consider two cases. In the first case, suppose the line PG intersects the invariant line l at point, say, L (see Figure 1.8(a)). Then the image of the line PG must be the line through the image point of G, G', and invariant point L. That is, the image of line PG is line $G'L$. So the image of point P, P', must also lie on line $G'L$. Since P also lies on line OP, then point P must be the intersection of lines OP and $G'L$. In the event that OP and $G'L$ are parallel, the point P has no image (since there is *no intersection* of parallel lines). In the second case, suppose the line PG is parallel to line l (then the invariant point L as used in the first case does not exist). Choose a point G_1 such that line G_1G is not parallel to l. Then the construction of the first case can be applied to find the image G_1' of G_1 (so point G replaces point O, and point G_1 replaces point P of the first case; also, line GG_1 replaces line OP of the first case). Applying the first case again with G_1 in place of point G, and point G_1' in place of G', we can find the image of P.

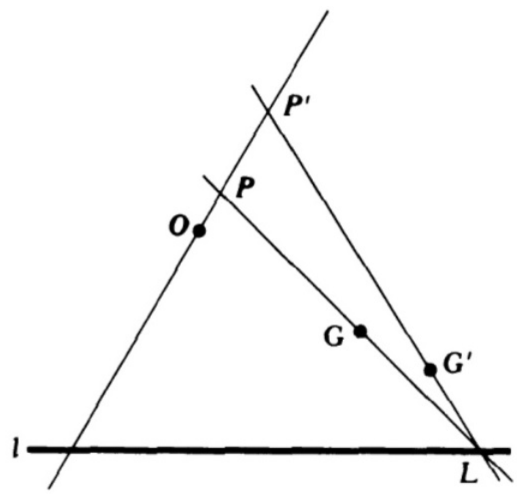


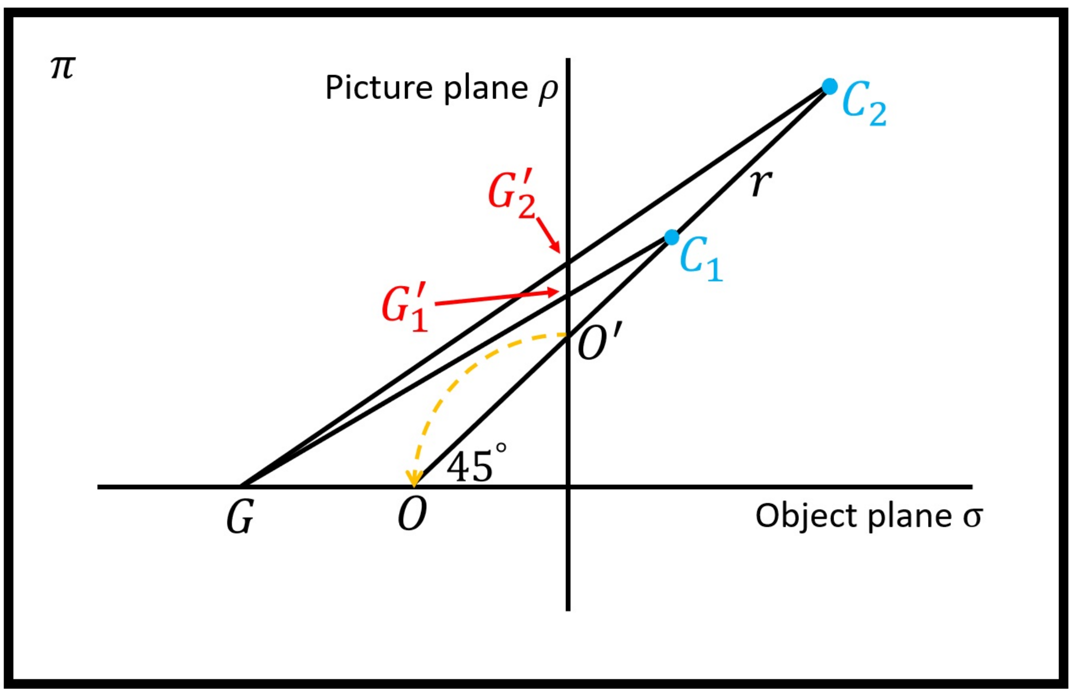
Fig. 1.8(a) The construction of the image of a point under a plane perspective.

Definition. The transformation defined in the two cases of Note 1.3.C (and hence projection followed by rabattement) is *plane perspective.* The line of invariant points l is the *axis* of the transformation, the invariant point O (not on line l) is the *center* of the transformation.

Note. We can now express the result of Note 1.3.B as follows.

Theorem 1.3.A. Let p be a line in the object plane through center O of a plane perspective transformation T. Then p is invariant under T.

Note. We see from Note 1.3.C that to uniquely determine a plane perspective, in addition to the axis l and center O, we need to know one additional point G and its image G'. We see this to be intuitively true if we consider a modified version of Figure 1.7. Invariant point O is determined by the viewing point C and the $45°$ line r. However, different viewing points can lie on line r and produce the same invariant point O (as C_1 and C_2 in the figure below). But then for point $G \neq O$, these different viewing points determine different images, say G'_1 and G'_2. However, if we know the image of G then we can determine the viewing point C and then the plane perspective is uniquely determined. We summarize this in the next theorem.



Theorem 1.3.B. A plane perspective transformation T is uniquely determined by its center O, axis l, and a pair of points (G, G') where G is not invariant and G' is the image of G under T.

Note 1.3.D. We now identity the points P which have no image under a plane perspective (as was a possibility in the first case discussed in Note 1.3.C). We saw in Note 1.3.C that point P has no image if and only if the line $G'L$ is parallel to the line OP in Figure 1.8. Let p be an arbitrary line through invariant point O on which a point P with no image lies. Through G' draw a line parallel to p and let L be the point where lines p and l intersect. See the figure below (on the left, the points O and P are on the same side of l and on the right they are on opposite sides of l; Wiley omits a figure covering the right configuration). Now point P lies on both line p and line LG. This is how such a point P can be found starting with a line through O (assuming this line actually contains a point with no image; we'll see that a line through O parallel to the invariant line l contains no such points) and the given pair of points (G, G') in the definition of the plane perspective. Since line $G'L$ is parallel to line OP (i.e., line p), then P has no image, as in Note 1.3.C.

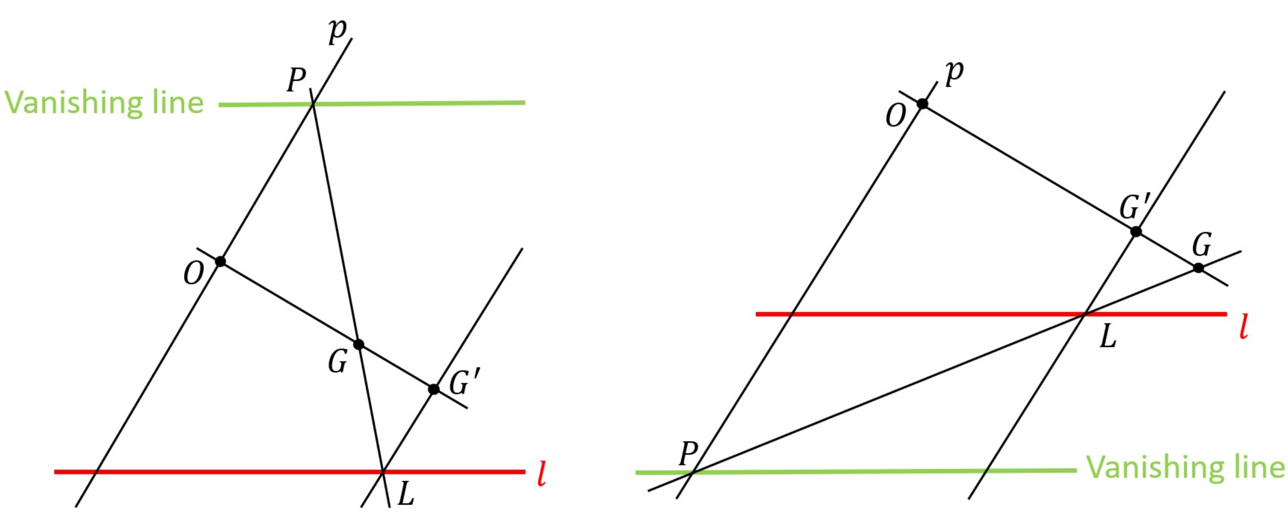


Figure 1.8(b) Modified. The vanishing line of the plane perspective.

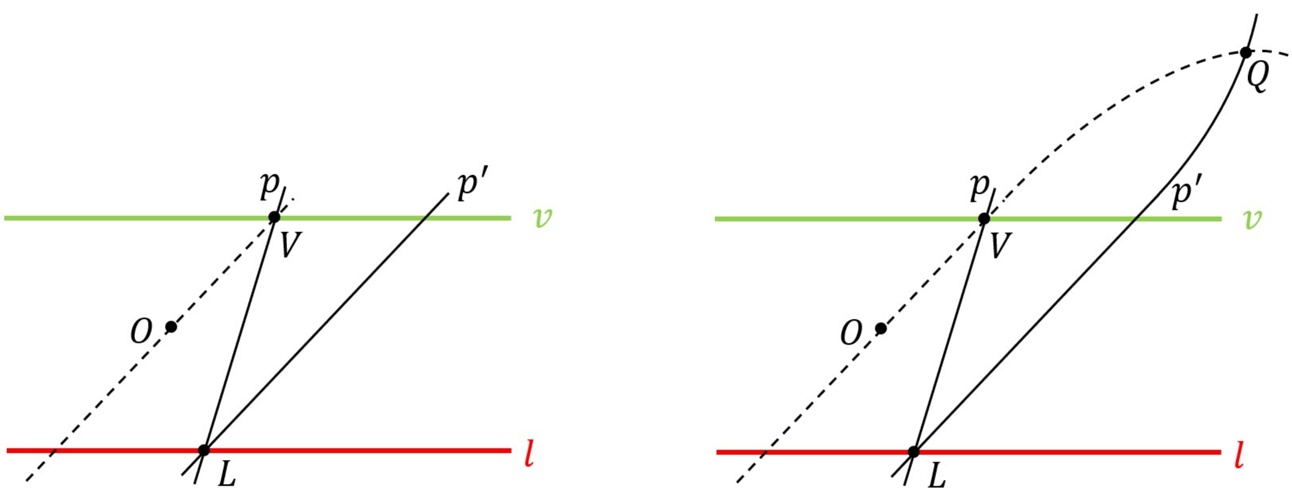
Notice that triangles $\triangle PGO$ and $\triangle LGG'$ are similar (because line LG' is parallel to line OP, and PL and OG' are transversals; all three angles corresponding angles are hence equal). Therefore, using parentheses to indicate lengths of line segments, we have $(PG)/(LG) = (GO)/(GG')$. Since each of the points O, G, and G' are fixed (they are part of the definition of the plane perspective), then $(GO)/(GG')$ is constant, and hence $(PG)/(LG)$ is also constant. Now $(PG)/(LG)$ is independent of line p (though it is dependent on point P). We now have from "elementary geometry" (as Wiley puts it on page 14; we accept this without a detailed proof) that if P and L are points collinear with a fixed point G such that $(PG)/(LG)$ is a constant, and if L varies along a line l, then P varies along a line parallel to l. That is, all points P which have no image under a plane perspective lie on a line parallel to line l.

Definition. The line of all points which have no image under the plane perspective transformation is the *vanishing line* of the plane perspective.

Note. The vanishing line of the perspective transformation (seen in the previous section; notice Figure 1.3) has little to do with the vanishing line of the plane perspective transformation here. Both lines are parallel to line l, and hence are parallel to each other, but the similarities end with this. In particular, the vanishing line of the plane perspective is not the line onto which the vanishing line in the picture plane is rotated in the 90° rotation of rabattement (see Exercise 1.3.3). We now state and prove some results that illustrate the importance of the vanishing line of a plane perspective.

Theorem 1.3.1. Let T be a plane perspective transformation with axis l, center O, and vanishing line v. Then the image of an arbitrary line p meeting l in a point L and meeting v in a point V is the line p' which contains L and is parallel to line OV. (See the figure below; we'll see that the proof is independent of whether O and v are on the same side of l, or not.)

Proof. Since L is an invariant point under T, then the image p' of given line p must contain point L. ASSUME p' intersects line OV, say at point Q. Then Q lies on line OV. Line OV is invariant under T by Theorem 1.3.A. Since V lies on both line OV and line p, and Q lies on the image of line OV (i.e., line OV itself) and Q lies on line p', then point Q must be the image of point V (two distinct lines can only share one point, of course). But this is a CONTRADICTION to the fact that V lies on the vanishing line and hence, by definition, has no image under T. So the assumption that lines p' and OV intersect is false. That is, lines p' and OV are parallel, as claimed. ∎

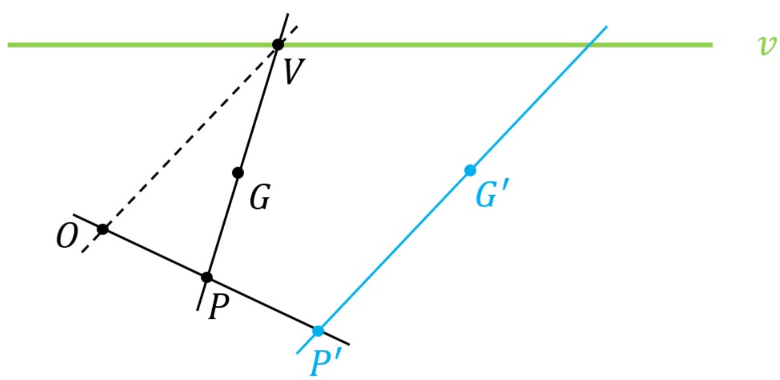


The claimed configuration of OV and p' (left), and the assumed intersection (right).

Corollary 1.3.1. In any plane perspective transformation, the image of the family of lines which pass through a point on the vanishing line of the perspective is a family of parallel lines.

Proof. With V as the point on the vanishing line through which the family of lines pass, we see by Theorem 1.3.1 that all lines in the family of lines are parallel to line OV and hence are parallel to each other, as claimed. ∎

Theorem 1.3.2. Let T be a plane perspective transformation with center O and vanishing line v, let G' be the image of a particular noninvariant point G under T, let P be an arbitrary point, and let V be the point of intersection of lines v and PG. Then the image of P under T, P', is the point common to line OP and the line through G' which is parallel to line OV.



Note. In Theorem 1.3.2, notice that we cannot have $P = O$ since we need to refer to the line OP. We cannot have G on line v since this would require that $G = V$, by the construction of V, and $G = G'$ since the points of v are invariant. Then the line through $G' = G = V$ and parallel to line OV would be line OV itself. The point common to line OP and the line through G' which is parallel to line

OV would then be point O, so that the image of P is O and, since P is arbitrary, we must have all points mapped to O. This represents the degenerate case where the viewing point C lies in the picture plane (see Figure 1.7 above, for example). These reasons are why we need to take G as an invariant point in Theorem 1.3.2.

Proof of Theorem 1.3.2. Since the point P is on line PG, then its image P' lies on the image of PG. By Theorem 1.3.1, the image of PG is the line through G' which is parallel to line OV (with line PG playing the role of line p in Theorem 1.3.1). Since line OP is fixed by plane perspective transformation T by Theorem 1.3.A, the the image of P, P', must also lie on line OP. So P' lies both on line PG (which is parallel to line OV) and on line OP. That is, the image of P is the point common to line OP and the line through G' which is parallel to line OV, as claimed. ∎

Note. We now see by Theorem 1.3.2 that if we are given the center, the vanishing line, and a pair of points (G, G') of a plane perspective transformation T, (where G' is the image of G under T), then we can determine the image of an arbitrary point P. We now summarize this in a corollary.

Corollary 1.3.B. A plane perspective transformation T is uniquely determined by its center O, vanishing line v, and a pair of points (G, G') where G is not invariant and G' is the image of G under T.