1.4. Plane Constructions

Note. In this section, we consider the images of certain given configurations under a plane perspective transformation. We know that points transform to points, and lines transform to lines, so we might expect that transformations of polygons is straightforward. We explore quadrilaterals in some detail, and also consider transformations of circles. We'll see that transformations of circles yield conic sections; the conic section that results depends on the circles position relative to the vanishing line.

Note 1.4.A. Consider the quadrilateral $ABCD$. We will transform it into a parallelogram. We first consider the case where no pair of opposite sides of the quadrilateral are parallel. Let V_1 be the point of intersection of opposite sides AB and CD (where we extend the line segments to infinite lines so that we can determine the point of intersection), and let V_2 be the intersection of opposite sides AD and BC. To create the plane perspective transformation that will map $ABCD$ to a parallelogram, we first introduce the vanishing line as line V_1V_2. See Figure 1.9. By Corollary 1.3.1 the images of AB and CD under any plane perspective with V_1V_2 as its vanishing line will be parallel, and the images of AD and BC under the same plane perspective will be parallel. This yields a desired plane perspective transformation. If we also know the center O of the transformation, then by Theorem 1.3.1 we know that the images of AB and CD will be lines parallel to line OV_1 and the images of AD and BC will be lines parallel to line OV_2. This is reflected in Figure 1.9. If we assign the location of the image A' of point A (we are free to choose a pair of points (G, G') where G' is the image of G under the plane perspective; see Corollary 1.3.B), the we can draw (infinite) lines $A'B'$ and $A'D'$ using the fact that they are parallel to lines OV_1 and OV_2, respectively (though we haven't yet found the precise locations of B' and D' yet). Now point B is common to lines AB and OB, so its image is common to lines $A'B'$ and $O'B' = OB'$ (since O is invariant). Also, line OB is the same as line $O'B'$ by Theorem 1.3.A. That is, we can determine the precise location of B' as the intersection of lines $A'B'$ and OB. Similarly, point D' can be precisely located as the intersection of line $A'D'$ and OD. Point C' can then be found through a similar argument, or based on the fact that we now know that $A'B'C'D'$ is parallelogram.

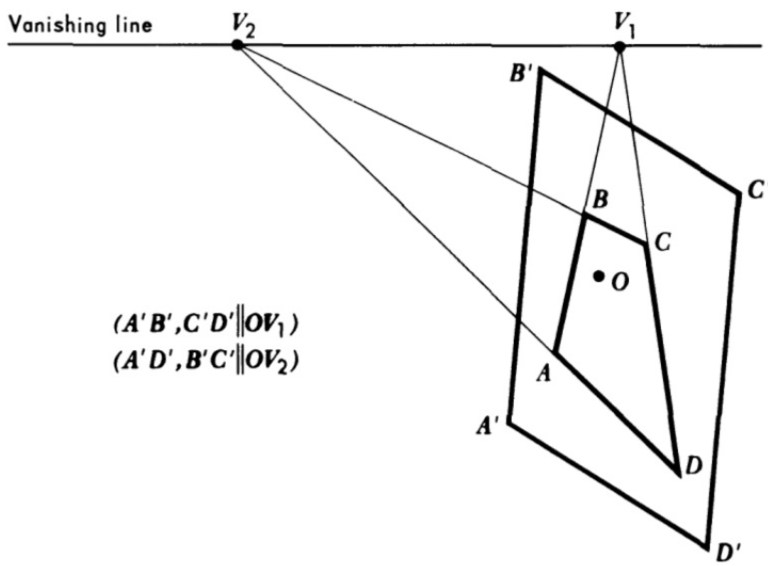
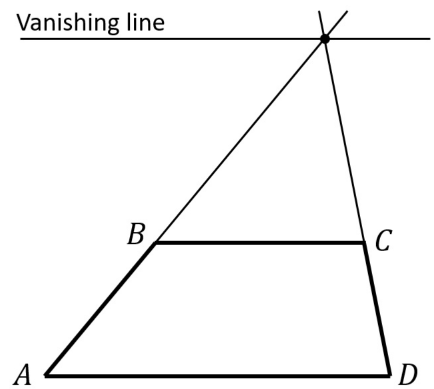


Fig. 1.9 The transformation of a general quadrilateral into a parallelogram.

If exactly two sides of $ABCD$ are parallel, say sides AB and CD, then we can take the vanishing line as a line parallel to sides AB and CD and passing through the point of intersection V of sides AD and BC (see the figure below). As in the previous case, we will have $A'D'$ and $B'C'$ parallel by Corollary 1.3.1, and lines $A'B'$ and $C'D'$ are parallel, as argued in Note 1.2.D. Again, there is a plane perspective transformation mapping $ABCD$ to a parallelogram. Of course, if $ABCD$ is already a parallelogram then we can simple take the identity transformation.



Note 1.4.B. Now suppose we want quadrilateral $ABCD$ to be transformed by the plane perspective transformation onto a rectangle. Then we need, for example, angle $\angle C'D'A'$ to be a right angle. Since $C'D'$ is parallel to OV_1, and $A'D'$ is parallel to OV_2 (as shown in Note 1.4.A), then $\angle C'D'A' \cong \angle V_1OV_2$. So we need to choose center point O of the transformation such that $\angle V_1OV_2$ is a right angle. We had no restriction on O in Note 1.4.A, but here we see that O must be on a circle with segment $\overline{V_1V_2}$ as its diameter (since an angle inscribed in a semicircle is a right angle). See Figure 1.10 below. Now we know the image under the transformation is a parallelogram (by the choice of the vanishing line in Note 1.4.A), so if one angle is a right angle then so are the other three and the image is a rectangle.

Note 1.4.C. We still have some liberties in the choice of the center O of the plane perspective transformation that takes quadrilateral $ABCD$ to rectangle $A'B'C'D'$. we now use the freedom to choose O is such a way that $A'B'C'D'$ is a square. Now a rectangle is a square if and only if its diagonals are perpendicular (a geometric fact that we accept without proof). So we keep O on the semicircle with diameter $\overline{V_1V_2}$, as in note 1.4.B. But we also want the diagonals $A'C'$ and $B'D'$ to be perpendicular. Let V_3 be the point of intersection of diagonal AC with the vanishing line, and let V_4 be the point of intersection of diagonal BD with the vanishing line (if either diagonal is parallel to the vanishing line, we can just shift point O on the semicircle of Figure 1.10 a little so that this is no longer the case). As in Note 1.4.B, we want the center O of the perspective transformation to also lie on a semicircle with diameter $\overline{V_3V_4}$. That is, we choose O as one of the two points of intersection of the two relevant circles. see Figure 1.11. Notice that the ordering of the edges and diagonals of the quadrilateral insure that points V_1, V_2, V_3, V_4 appear on the vanishing line in an order that alternates between the two circles (so the circles really do intersect in two points).

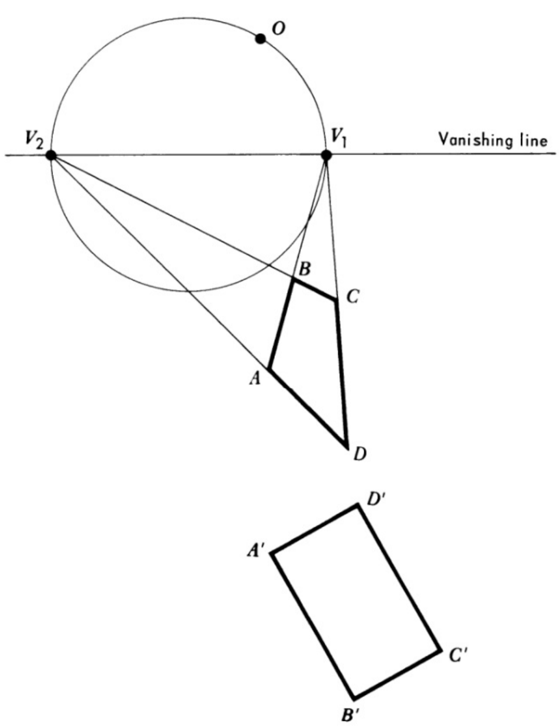


Fig. 1.10 The transformation of a quadrilateral into a rectangle.

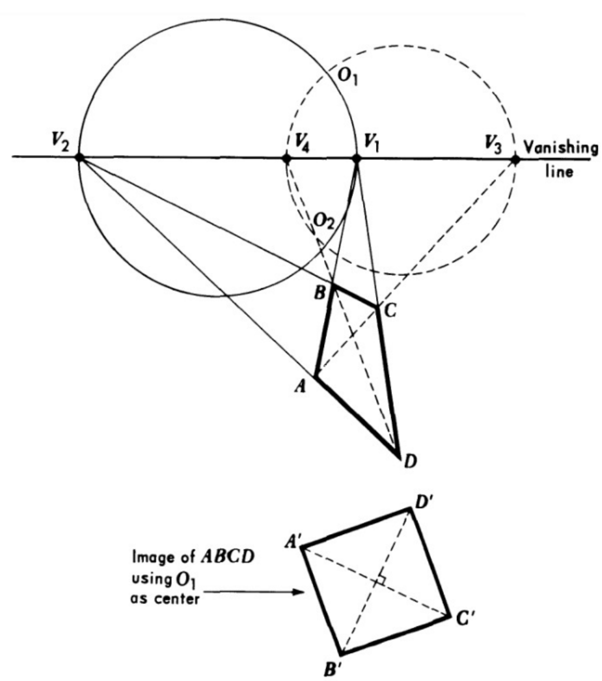


Fig. 1.11 The transformation of a quadrilateral into a square.

Note 1.4.D. We now turn our attention to the images of various circles under the plane perspective transformation. We know from Corollary 1.3.B that a plane perspective is determined from the center O, the vanishing line v, and the image G' of some point G. A perspective transformation, as discussed in Section 1.2. The Elements of Perspective, maps conic sections in the object plane to conic sections in the picture plane (see Note 1.2.G in Section 1.2. The Elements of Perspective, and Exercise 1.2.C(a)). Since a plane perspective is a perspective transformation followed by a 90° rotation of the picture plane onto the object plane (that is a plane perspective followed by rabattement), then a plane perspective transformation also maps conic sections to conic sections.

Note 1.4.E. Consider the image of a circle Γ under a plane perspective. First, suppose that the circle does not intersect the vanishing line v. See Figure 1.12.

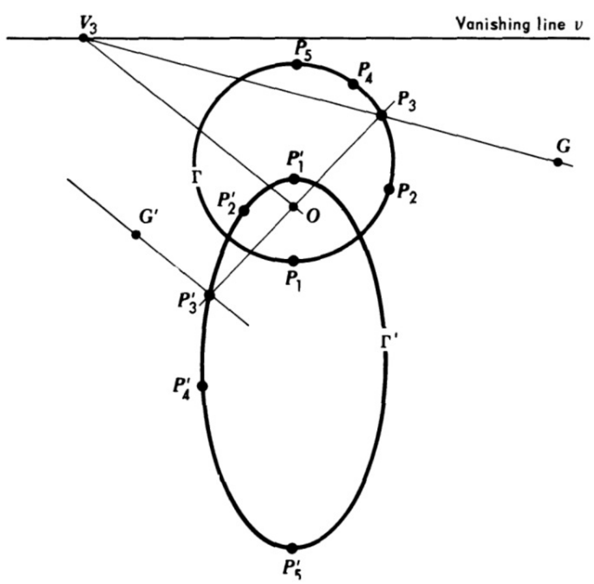


Fig. 1.12 The transformation of a circle into an ellipse.

Since the only points that do not have images are those on the vanishing line, then each point of Γ has an image. This implies that the image of Γ is bounded (we could use some details here; continuity of the plane perspective on the connected components of the plane minus the vanishing line seems necessary). The circle is mapped to some conic section by Note 1.4.D. The only bounded conic section is an ellipse, so that the circle must be mapped to an ellipse (but remember that a circle is a special case of an ellipse, so this is a possible image also). Figure 1.12 illustrates the projection of point P_3 to point P_3' using O, v, G, and G' and Theorem 1.3.2; the other points are similarly projected.

Note 1.4.F. Consider again the image of a circle Γ under a plane perspective. First, suppose that the circle intersects the vanishing line v in two points V_1 and V_2. See Figure 1.13. These points have no images since they lie on v.

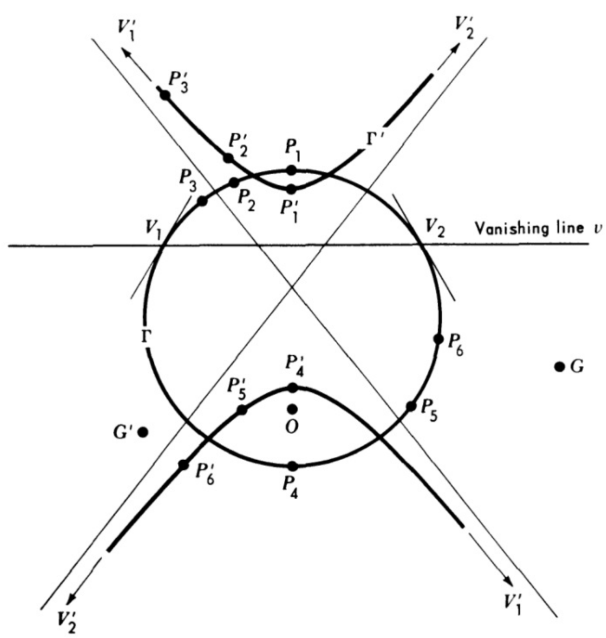


Fig. 1.13 The transformation of a circle into a hyperbola.

Now the tangent to the circle at V_1 intersect the circle at only one point, namely V_1. By Theorem 1.3.2 this tangent line is transformed to a line parallel to OV_1 and which intersects the circle in no other points (since the tangent line intersects the circle in no other points). Similarly, the tangent to the circle at V_2 transforms to a line parallel to OV_2 which intersects the circle in no other points. Any other line passing through V_1 (or V_2), except the vanishing line v, intersects the circle in a second point and this second point is not on the vanishing line. Such a line is transformed to a line parallel to OV_1 (or OV_2; by Theorem 1.3.2 again) which meets the image of the circle in a single point. We know by Note 1.4.D that the image of the circle Γ is a conic section, so the image must be a hyperbola with the images of the tangents to the circle at V_1 and V_2 and the asymptotes.

Note 1.4.G. Consider again the image of a circle Γ under a plane perspective. This time suppose that the circle intersects the vanishing line v in exactly one point V. In this case, v must be tangent to circle Γ. See Figure 1.14.

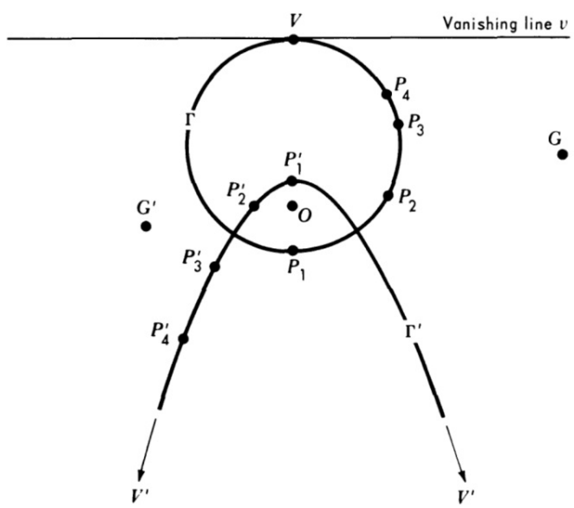


Fig. 1.14 The transformation of a circle into a parabola.

Now any line through V, other than the vanishing line v, intersects circle Γ at one other point. Such a line is transformed to a line which is parallel to line OV (by Theorem 1.3.2) and which intersects the image of Γ in a single point. We know by Note 1.4.D that the image of the circle Γ is a conic section, so the image must be a parabola with an axis parallel to OV. The vertex of the parabola will be the intersection of line OV with Γ.

Note 1.4.H. Wiley gives an intuitive interpretation of the vanishing line and the images of a circle. "[I]t appears that the points of the vanishing line are transformed into points which are infinitely far away, i.e., points which lie on a 'line at infinity'. . . " (as Wiley says on page 23). Thinking of these points as "transformed to infinity" makes the results of Figures 1.12, 1.13, and 1.14 intuitively plausible. In these figures, points near the vanishing line v and below v are mapped to points far from v and below v. Points near the vanishing line v and above v are mapped to points far from v and above v. For example, think of the transformation of Figure 1.13 as mapping (1) the part of the circle below v to the lower branch of the hyperbola, (2) the part of the circle above v to the upper branch of the hyperbola, and (3) the two points V_1 and V_2 of the circle on v "to infinity."

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