6.2. The Axioms of Incidence and Connection

Note. In this section, we give the undefined terms, state six axioms, and state four theorems (the proofs of which are given as exercises). We discuss a duality relationship between lines and points (and incidence conditions).

Note. The necessity of undefined terms in Euclidean Geometry is discussed in Introduction to Modern Geometry (MATH 4157/5157); see my online notes from Introduction to Modern Geometry on Section 1.3. Axiomatic Systems. Three undefined terms in projective geometry (which were also undefined in Euclidean geometry) are "point," "line," and a relation referred to (variously) as "lies on," "passes through," "contains," or "is incident with." We model these ideas with the usual drawings. However, as Wylie states in this section, these drawings "are only reminders of more abstract concepts, and ultimately points and lines are, or can be thought of as, any objects whatsoever which have the properties the axioms attribute to points and lines."

Definition. A *projective geometry* consists of the undefined terms "points" and "lines" which satisfy the following axioms.

Axiom 1. For any two (distinct) points, there is at least one line which contains both points.

Axiom 2. For any two (distinct) points, there is at most one line which contains both points.

Axiom 3. For any two lines, there is at least one point which lies on both lines.

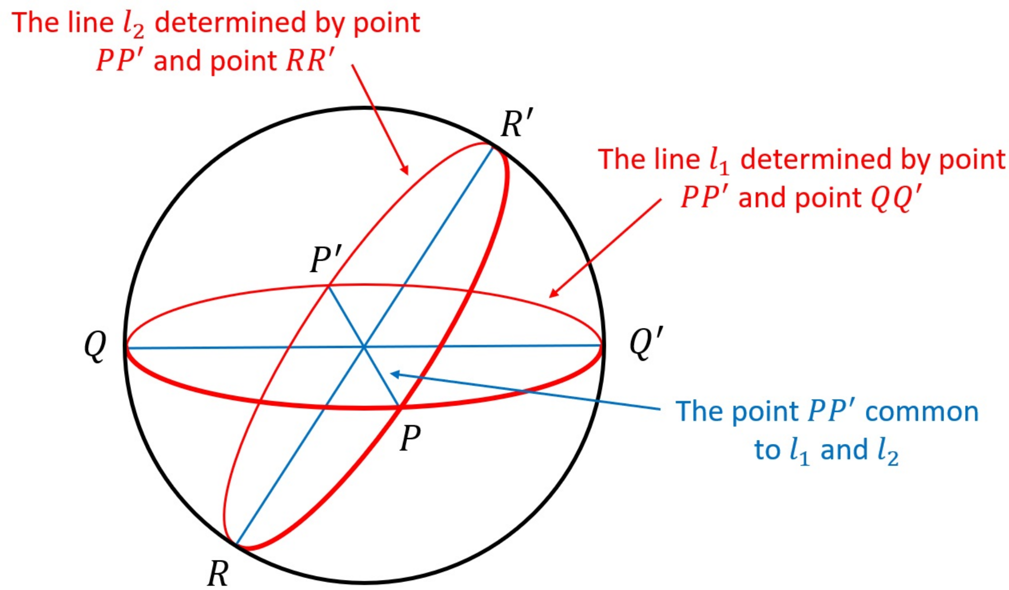
Axiom 4. Every line contains at least three (distinct) points.

Axiom 5. All points do not lie on the same line.

Axiom 6. There exists at least one line.

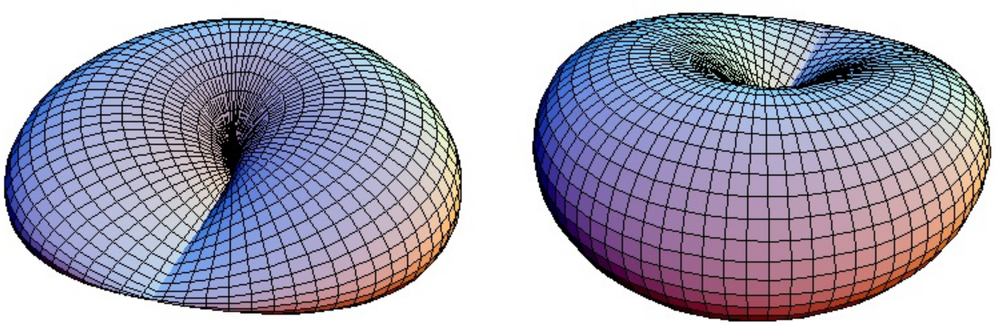
Note. Notice that Axioms 1 and 2 together imply that there is exactly one line containing any two points. We see that projective geometry is an example of non-Euclidean geometry because, by Axiom 3, no two lines are parallel in projective geometry (recall that in Euclidean geometry, two lines are *parallel* if they do not intersect).

Note 6.2.A. Two models of plane projective geometry are given in the transformation part of Axiomatic and Transformational Geometry (MATH 5330). One model uses the complex plane $\mathbb{C}$, and another uses $\mathbb{C}^3$; see my online notes for transformational geometry on Section 60. The Complex Projective Plane and Section 61. A Model for the Projective Plane, respectively. We can use a sphere to produce an easily visualized model (after all, we live on a sphere!). Let S be a sphere in Euclidean 3-space E_3. We give the unintuitive interpretation of points and lines as follows: a *point* is a diameter of S and a *line* is a great circle of S. A line *contains* a point if and only if the point (i.e., diameter of S) is a diameter of the line (i.e., great circle of S). This is called the *spherical model* of projective geometry. See the figure below, based on Wylie's Figure 6.1. The six axioms of projective geometry are easily verified to hold for this model. In fact, the spherical model of projective geometry is isomorphic to the extended Euclidean plane, E_2^+ given in Section 2.3, "Ideal Points and the Ideal Line" (see page 55). We leave this isomorphism claim to Exercise 6.2.5.



Two points on S form a diameter $\overline{PP'}$ if the pair of points (P, P') are "diametrically opposite" on S. As Wiley puts it (see page 249): "...we may consider either P or P' itself as a representative of a point of our system, since P determines P', and conversely." A more mathematically rigorous approach is to set up an equivalence relation $\cong$ between the points on S such that $P \cong P'$ if and only if P and P' are diametrically opposite on S. We then take the *points* in the model to be equivalence classes of points under $\cong$. This is explained in much more detail in my online notes for the algebraic topology component of Introduction to Topology (MATH 4357/5357) on Section 60. Fundamental Groups of Some Surfaces. You encountered equivalence relations in Mathematical Reasoning (MATH 3000; see my online notes for that class on Section 2.9. Set Decomposition: Partitions and

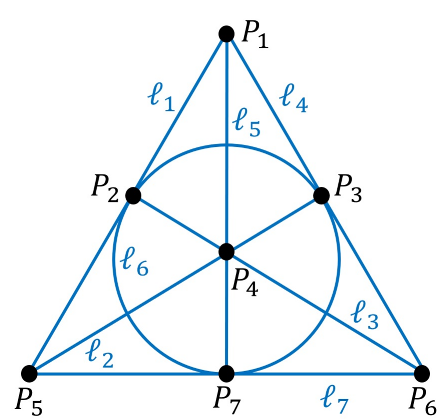
Relations and notice Definitions 2.55 and 2.57). You will see equivalence relations and equivalence classes in Introduction to Modern Algebra (MATH 4127/5127) where you make a group out of the equivalence classes (called a “factor group” or “quotient group”); see my online notes for that class on Section II.10. Cosets and the Theorem of Lagrange. We can also deal with the spherical model by only considering a hemisphere of the sphere, along with half of the points on the boundary of the hemisphere (for every point on the boundary which is included, exclude the point diametrically opposite it). The *projective plane* then results by “sewing” together the boundary of the hemisphere where diametrically opposite points on the boundary are connected. This cannot be done in three dimensions because the sewing operation requires the boundary of the hemisphere to “cross inside itself.” A common three-deimsnional representation of the projetive plane is the following image:



An image of the projective plane from the Wikipedia webpage on the Real Projective Plane (accessed 10/8/2023).

Note 6.2.B. We now address the consistency of the six axioms of projective geometry. The ideas of *absolute consistence* and *relative consistence* are addressed in Introduction to Modern Geometry (MATH 4157/5157); see my online notes for that class (which are also based on a book written by C. R. Wiley) on Section 1.4. Consistency. The spherical model given in Note 6.2.A establishes the relative consistency of projective geometry. Since the spherical model is based on Euclidean geometry, we have that *if* Euclidean geometry is consistent *then* projective geometry is consistent (we have to settle for relative versus absolute consistency, since there is no absolute test for consistency of a set of axioms). Another model is based on arithmetic and the algebraic representation, Π_2, of the extended Euclidean plane (see Section 2.3, "Ideal Points and the Ideal Line," and page 55 of the book for the details on this example). A simpler example is a finite projective plane. Let the *points* by the elements of the set $\{P_1, P_2, P_3, P_4, P_5, P_6\}$, let the *line* be the following sets of size three: $\ell_1 = \{P_1, P_2, P_3\}$, $\ell_2 = \{P_3, P_4, P_5\}$, $\ell_3 = \{P_2, P_4, P_6\}$, $\ell_4 = \{P_1, P_3, P_6\}$, $\ell_5 = \{P_1, P_4, P_7\}$, $\ell_6 = \{P_2, P_3, P_7\}$, and $\ell_7 = \{P_5, P_6, P_7\}$. A line *contains* a point if the point is an element of the line. The model can be summarized in the following table and visual representation of lines and points they contain:

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7
P_1	P_3	P_2	P_1	P_1	P_2	P_5
P_2	P_4	P_4	P_3	P_4	P_3	P_6
P_5	P_5	P_6	P_6	P_7	P_7	P_7



The figure above is called the "Fano plane" and you see it in finite geometry or graph theory. For example, see my online notes for graph Theory 1 (MATH 5450) on

Section 1.3. Graphs Arising from Other Structures (notice Figure 1.15(a)). Wiley also gives examples of finite projective plans with 13 points (and 13 lines), and with 21 points (and 21 lines). So the relative consistence of projective geometry based on Axioms 1 through 6 is well established. In Exercise 6.2.2, a finite projective geometry for which there exists a line containing n points is considered, and the following are to be proved:

(a) Every line contains exactly n points.

(b) Every point has exactly n lines passing though it.

(c) The system contains exactly $n^2 - n + 1$ points.

(d) The system contains exactly $n^2 - n + 1$ lines.

Note 6.2.C. We now state four theorems, the proofs of which are rather easy and are to be given in Exercise 6.2.1.

Theorem 1. There exists at least one point.

Theorem 2. For any two lines, there is at most one point which lies on both lines.

Theorem 3. All lines do not through the same point.

Theorem 4. Every point lies on at least three lines.

Note 6.2.D. Let's compare Axiom 4 and Theorem 4. Axiom 4 states that "Every line contains at least three points," and Theorem 4 states that "Every point lies on at least three lines." This illustrates the *Principle of Duality* between the roles of lines and points. Notice that if we interchange the words "line" and "point," and interchange the relationships of "contains" and "lies on," then one statement becomes the other. In fact, this holds for Axioms 1 through 6 and Theorems 1 through 4, with the following dualities:

Axiom 1 ↔ Axiom 3	Theorem 1 ↔ Axiom 6
Axiom 2 ↔ Theorem 2	Theorem 2 ↔ Axiom 2
Axiom 3 ↔ Axiom 1	Theorem 3 ↔ Axiom 5
Axiom 4 ↔ Theorem 4	Theorem 4 ↔ Axiom 4
Axiom 5 ↔ Theorem 3	
Axiom 6 ↔ Theorem 1	

Since the dual of each axiom also holds, then for any theorem provable from Axioms 1 through 6 also has a provable dual theorem. We simply replace each axiom or theorem in the proof with its dual. That is, any claim holding in projective geometry concerning points and lines, and based on Axioms 1 through 6 yields another true claim which results from the original one by interchanging the term "point" and "line" and interchanging the relationships of "contains" and "lies on." If we add additional axioms, then we may lose the Principle of Duality. So if we want to add additional axioms and maintain the Principle of Duality (both of which we want), then for each new axiom added we must also either add the dual of the axiom or prove that the dual is a theorem in the new axiomatic system. Of course, adding new axioms requires that we address consistency of the new axiomatic system.

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