1.2. Inductive and Deductive Reasoning

Note. In this section, we give Wylie's definitions of inductive and deductive reasoning. We discuss the advantages and restrictions of each. We will directly quote Wylie several times (updating his use of pronouns in places!).

"Definition." Induction is the process of inferring general properties, relations, or laws from particular instances which have been observed. Deduction is the process of reasoning to particular conclusions from general principles that have been accepted as the starting point of an argument. (See page 2 of the Wylie's book.)

Note. Wylie gives some warnings about the use of induction. (We should note that we are *not* talking about Mathematical Induction here. The Principle of Mathematical Induction is an axiom in the development of the natural numbers; see my online notes for Modern Algebra 1 [MATH 5410] on Chapter 0. Introduction: Prerequisites and Preliminaries, in particular Axiom 5 of Peano and Theorem 0.6.1). If we consider many, many cases of some infinite-case situation, then even if the many, many (but finite) cases imply a certain behavior then we still cannot conclude that all of the cases have that behavior. Consider the expression $n^2 - n + 17$ where n is a natural number, $n \in \mathbb{N}$. The values of the expression for $n = 1, 2, \ldots, 10$ are:

n	$n^2 - n + 17$	n	$n^2 - n + 17$
1	17	6	47
2	19	7	59
3	23	8	73
4	29	9	89
5	37	10	107

Some conjectures based on these values include:

(1) $n^2 - n + 17$ is odd for all $n \in \mathbb{N}$,

(2) $n^2 - n + 17$ is a number which ends in 3, 7, or 9 for all $n \in \mathbb{N}$, and

(3) $n^2 - n + 17$ is a prime number (as are each of the values of the expression given above) for all $n \in \mathbb{N}$.

This last conjecture couldn't possibly be true; there is no known formula for generating primes! Continuing the table:

n	$n^2 - n + 17$	n	$n^2 - n + 17$
11	127	16	257
12	149	17	289
13	173		
14	199		
15	227		

Now 289 is not prime, since $17^2 = 289$. On the other hand, as is to be shown Exercise 1.2.1, conjectures (1) and (2) actually are true.

Note 1.2.A. Now we turn to the shortcomings of deduction. In deduction, we prove things but they are always of the form "*If* something is true, *then* something else is true." So first we need the initial statements from which we will deduce the other statements; that is, we need the axioms on which we base everything. The desirable properties of this collection of axioms are explored in the reminder of this chapter. In choosing the axioms, induction is useful. Induction also can inspire the axioms we attempt to prove. "The proofs of theorems are the fruits of deductive reasoning, but theorems themselves are the fruits of induction, of intuition, of creative insight habitually examining every special case for suggestions of more general relations. . . . without skill in both induction and deduction, that is, without both intuition and a clear feeling for proof, no student of mathematics is more than marginally prepared for [their] work." (As Wylie says on pages 4 and 5.) Exercises 1.2.8 through 1.2.14 describe certain geometric "experiments" which are to inductively inspire a general conclusion. For this class, we are certainly indebted to induction for the axioms, since much of the pure mathematics of geometry is inspired by the physical world and our attempts to model it.

Note. Wylie dips into the philosophy of mathematics on page 5: "And though mathematics is ultimately a construction of the mind alone and exists only as a magnificent collection of ideas, from earliest times down to the present day it has been stimulated and inspired and enriched by contact with the external world. . . . Many of its concepts are abstractions from the common experience of all [people]. . . . Surely, without contact with the external world, mathematics, if it existed at all, would be vastly different from what it is today." A well-known paper which discusses the intimate relationship between mathematics and "the external world" is Eugene Wigner's "The unreasonable effectiveness of mathematics in the natural sciences, Richard Courant lecture in mathematical sciences delivered at New York University, May 11, 1959," *Communications on Pure and Applied Mathematics* **13**: 1-14 (1960). A copy is available online on Andrew Ranicki's Homepage (accessed 8/31/2021).

Note. Wylie next addresses the pure, abstract, or "useless" nature of mathematics. He offers two defenses of pure mathematics, the first of which is based on an artistic appeal: "In the first place, all mathematics worthy of the name is pure or abstract or even, in a certain sense, useless, just as poetry and music and painting and sculpture are useless except as they bring satisfaction to those who create and to those who enjoy." His second argument is more pragmatic: "Then in the second place, by a remarkable coincidence which is almost completely responsible for the existence of all present-day science and technology and which has no counterpart in the arts, even the most abstract parts of mathematics have turned out again and again to be highly 'practical,' as science, becoming ever more sophisticated, finds that it needs mathematical tools of greater and greater refinement." See page 5.

Revised: 9/4/2021