1.3. Axiomatic Systems

Note. In this section, we discuss the basic parts of an axiomatic system and give explanations as to why undefined terms and unproved axioms are necessary. We briefly discuss the properties of consistence, independence, completeness, and categoricalness.

Note. An axiomatic system consists of:

1. A set of undefined terms which forms the basis of the necessary technical vocabulary,

2. A set of unproved initial assumptions,

3. The laws of logic, and

4. The body of theorems, expressing properties of the undefined objects, which are derived from the axioms by the laws of logic.

Note 1.3.A. Let's discuss the set of undefined terms. Since we can only define new terms by means of previously defined terms, then we must have some starting-point of terms (just as we must have a collection of initial assumptions). Trying to define every term either leads to some first undefined word, or to some circular collection of definitions that loops back to an already-defined term. Wylie describes this by looking up a term in a dictionary, and then looking up the terms in the definition, etc., until the path takes us back to the original term (see page 9 of the text book). This gives us a vocabulary consisting of the starting-point undefined terms, other terms defined in terms of these, and "the nontechnical vocabulary of everyday discourse which, of course, we implicitly assume to be available" (as Wylie puts it; see also page 9). Examples of this everyday discourse include ideas like "none," "some," "or," and "all" (and "and"!). Examples of terms we leave undefined in Euclidean geometry are those of "point" and "line." Euclid does not take this approach himself, and instead gives the following useless definitions in Book I:

Definition 1. A point is that which has no part.

Definition 2. A line is breadthless length.

Note. We mentioned the unproved initial assumptions in the previous section (see Note 1.2.A). With deductive reasoning, we deduce statements of the form "If A then B." So we are lead to an infinite chain of deductions of this sort: "If A then B, if B then C, if C then D, etc.," and we never establish claim A. Alternatively, the deductions wind back around to claim A in which we have committed circular reasoning! So we see that we cannot prove every claim and certain statements must be taken and unproven claims. In contemporary terminology, this foundational claims are called *axioms* or *postulates*.

Note. It is sometimes said that axioms are statements that are so obvious that they do not need proof. This is a tempting concept in something as intuitive as Euclidean geometry. However, some axiomatic are counterintuitive, such as the axioms of non-Euclidean geometry. For example, the parallel postulate for hyperbolic geometry states that for a line L and a point P not on line L, there is more than one line (in fact, infinitely many) through point P that is parallel to line L. In fact, attempts to prove that this is NOT the case, lead to the "discovery" of non-Euclidean geometry. For details, see my online notes on Hyperbolic Geometry; notice the Hyperbolic Parallel Postulate, H5, where the postulate is stated in terms of the summit angles of a Saccheri quadrilateral. See also Postulate 17_H in "Section 4.2. The Parallel Postulate for Hyperbolic Geometry" of the text book, where this idea of parallels through a point not on a line is used.

Note. The choice of the axioms may be motivated by the "real world," but once the axioms are accepted, the real world is left behind and the manipulations of the mathematical objects becomes purely a work of the mind. Wylie observes (see page 10):

> "To abandon contact with the 'real' world in this fashion may seem foolish to those of a practical turn of mind, but it is highly practical. It allows the instruments of the mind to replace the instruments of the hand and eye in the study of the phenomena of original interest. And if the initial abstraction from the world of experience was made with appropriate care, the results deduced from the axioms by the laws of logic can be transported back into the 'real' world either as properties supported now by deduction as well as induction or often as new properties that were previously unknown."

Note. A system of axioms must be consistent. The property of independence is desirable, but not mandatory. Other desirable properties are completeness and categoricallness. The informal definitions of these terms are:

Definition. An axiomatic system is *consistent* if no axiom contradicts any other, and no two deductions from the axioms (these deductions are the theorems that follow from the axioms) contradict each other.

Definition. An axiomatic system is *independent* if no axiom can be deduced as a theorem from the other axioms (notice that this means that the axiomatic system is minimal).

Definition. An axiomatic system is *complete* if any meaningful statement involving the undefined terms and relations of the axiomatic system can be proved to be either true or false.

Definition. An axiomatic system is *categorical* if (informally put) all systems obtained by giving specific interpretations to the undefined terms of the abstract systems all essentially the same.

Note. In conclusion, we elaborate some on completeness and categoricalness. First, an example of a statement from geometry that is *not* meaningful is: "All points are parallel." Though points are part of geometry and the relation of "parallel" may or may not holds between lines, there is no concept of parallelness between points. So this statement is neither true nor false, but it meaningless. The idea of a categorical axiomatic system is from model theory. The claim here is that if two models satisfy the axiomatic system then the two models are "essentially" the same (the sameness would relate to some type of mapping, such as an isomorphism).

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