1.4. Consistency

Note. In this section, we informally define absolute consistency and relative consistency. We give an example of an absolutely consistent axiomatic system.

Note. In the previous section had the definition: "An axiomatic system is *consistent* if no axiom contradicts any other, and no two deductions from the axioms (these deductions are the theorems that follow from the axioms) contradict each other." Notice that this requires us to test *all* of the theorems that follow from the axioms, so one would expect that establishing the consistency of an axiomatic system to be quiet a chore! In practice consistency, when it *can* be established, is given by exhibiting a specific model whose elements and relations satisfy the axioms.

"Definition." An axiomatic system is *absolutely consistent* if a model of the system exists which involves objects and relations from the external ("real") world. If a model for the axiomatic system exists which involves objects and relations from another axiomatic system (such as Euclidean geometry or elementary arithmetic) then the system if *relatively consistent.*

Note. Since the external world cannot be inconsistent then an absolutely consistent axiomatic system must be consistent. A relatively consistent axiomatic system is consistent if the other axiomatic system to which it is relatively consistent is consistent (and if the other system is inconsistent then the original system is inconsistent).

Note. We now give a collection of six axioms involving two undefined objects. We will show the absolute consistency of the axiomatic system. The two undefined objects are x and y. An undefined relation is "belonging to." We take the axioms as:

A.1. If x_1 and x_2 are any two (distinct) x's, then there is at least one y belonging to both x_1 and x_2.

A.2. If x_1 and x_2 are any two x's, there is at most one y belonging to both x_1 and x_2.

A.3. If y_1 and y_2 are any two y's, there is at least one x belonging to both y_1 and y_2.

A.4. If y_1 is any y, there are at least three x's belonging to y_1.

A.5. If y_1 is any y, there is at least one x which does not belong to y_1.

A.6. There exists at least one y.

Note. We now give two models to show that this axiomatic system is consistent. We follow Wylie examples, but update the names and terminology in the first model. Suppose the x represents students and y represents academic clubs. We interpret "x belongs to y" to mean that student x is a member of club y. We interpret "y belongs to x" to mean that y is a club that has x as a member. Hence the relation "x belongs to y" holds if and only if "y belongs to x." We take the set of students as $X = \{$Amy, Berg, Cindy, Dunn, Ellen, Ford, Ginger$\}$ and the set of clubs as $Y = \{$Math, Sociology, Physics, Astronomy, Abstract Algebra, Engineering, History$\}$. In the following table, we have that x belongs to y if and only if there is an entry of B in the row determined by x and the column determined by y:

	Amy	Berg	Cindy	Dunn	Ellen	Ford	Ginger
Math	B	B	B				
Sociology	B			B	B		
Physics	B					B	B
Astronomy		B		B		B	
Abstract Algebra		B			B		B
Engineering			B	B			B
History			B		B	B	

It is easily verified that Axioms A.4, A.5, and A.6 are satisfied. To verify Axioms A.1 and A.2, we must consider every possible pair of students (of which there are $\binom{7}{2} = 21$) and confirm that there is one and only one club to which each pair of students belong. To verify Axiom A.3, we must consider every possible pair of clubs (of which there are $\binom{(}{7})(2) = 21$) and confirm that there is at least one student belonging to both clubs in the pair. The model, which we refer to as "Model 1," is a concrete, "real world" representation of the axiomatic system. There can be no inconsistency in Model 1, unless there is an inconsistency in real life. So the axiomatic system is absolutely consistent!

Note. We new give Model 2 that will also represent the axiomatic system above. This time, we take a more abstract approach. We represent x as a point in the plane and have the set of points as $X = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$. We then let y represent either one of the line segments in Figure 1.1 of P_1P_2, P_2P_3, P_1P_3, P_1P_6, P_2P_4 P_3P_5 or the circle $P_4P_5P_6$, which we denote as s_1, s_2, s_3, s_4, s_5, s_6, and c, respectively. So we have the set containing the y's as $Y = \{s_1, s_2, s_3, s_4, s_5, s_6, c\}$ (notice from the figure that each element of Y contains three points of X). This time, if point x is a point on segment (or circle) y then we have "x belongs to y" and "y belongs to x" (and conversely). We can again verify that this model satisfies Axioms A-1 through A.6 (and again confirm that the axiomatic system is absolutely consistent).

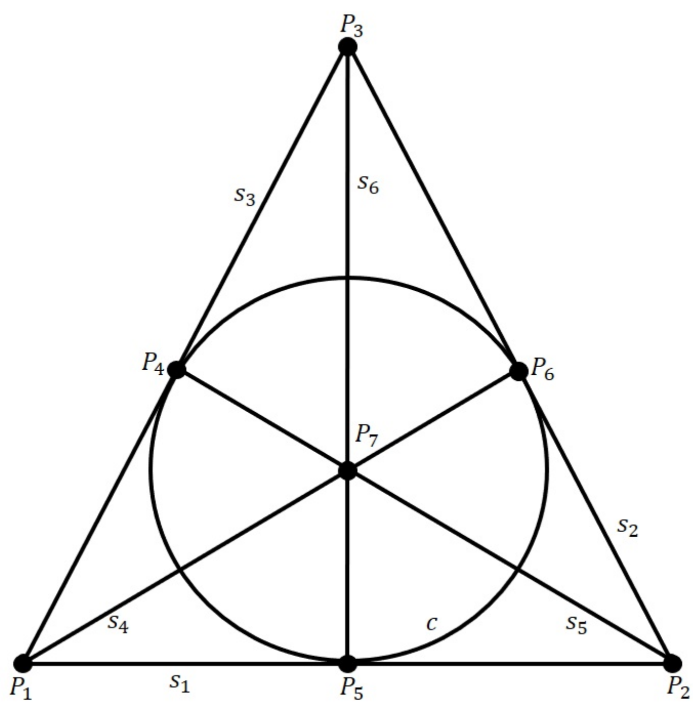

Figure 1.1. Model 2

Note. The configuration given n Figure 1.1 is called the *Fano plane*. It is the simplest example of a finite projective plane. We have the following definition from discrete math (see my online notes for Graph Theory 1 [MATH 5340] on Section 1.3. Graphs Arising from Other Structures; the definition of "finite projective plan" is from Exercise 1.3.13 of this section in Bondy and Murty's *Graph Theory*, Graduate Texts in Mathematics 244 (Springer, 2008)):

Definition. A *geometric configuration* $(P, \mathcal{L})$ consists of a finite set P of elements called *points* and a finite family $\mathcal{L}$ of subsets of P, called *lines*, with the property that at most one line contains any given pair of points. A *finite projective plane* is a geometric configuration $(P, \mathcal{L})$ in which:

(i) any two points lie on exactly one line,

(ii) any two lines meet in exactly one point,

(iii) there are four points, not three of which lie on a line.

In Exercise I.3.13(a) of Bondy and Murty's book, it is to be shown that for any finite projective plane $(P, \mathcal{L})$, there is an integer $n \geq 2$ such that $|P| = |\mathcal{L}| = n^2 + n + 1$ where each point lies on $n + 1$ lines, and each line contains $n + 1$ points. This integer is called the *order* of the finite projective plane. So the Fano plane is is a finite projective plane of order 2.

Note. We will see Model 1 and Model 2 again in Section 1.6. Completeness and Categoricalness, and we will explore finite projective planes in Section 1.7. Finite Geometries (where they are called "finite projective geometries).

Revised: 9/13/2021