1.5. Independence

Note. In this section, we show the axiomatic system given in Section 1.4 is independent by showing the consistency of several associated axiomatic systems. We also mention two historical examples when independence was originally overlooked.

Note. In the Section 1.3 we had the definition: "An axiomatic system is *independent* if no axiom can be deduced as a theorem from the other axioms (notice that this means that the axiomatic system is minimal)." An axiomatic system *must* be consistent (as described in the previous section), but independence is not an absolutely necessary property of an axiomatic system.

Note. In R.L. Devaney's *An Introduction to Chaotic Dynamical Systems* (Addison-Wesley, 1989) an "iterated function system" $f : J \to J$ is defined as *chaotic* if (1) f is topologically transitive, (2) f has sensitive dependence on initial conditions, and (3) periodic points of f are dense in J. Though not exactly an axiomatic system, this definition gives three conditions on an iterated function system. It was shown in 1992 that if an iterated function system is topologically transitive and has denseness of periodic points, then it necessarily has sensitive dependence on initial conditions; see J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey's "On Devaney's Definition of Chaos," *The American Mathematical Monthly*, **99**(4), 332–334 (1992); this is available online from JSTOR (accessed 9/19/2021). See also my online presentation A Mathematician Looks at Chaos given in the ETSU Physics Department Seminar in March 1997. So the dependence of one part of the definition on the other parts was not originally recognized.

Note. Though an independent axiomatic system is minimal, by starting with the such a system we have less assumptions to work with and so proofs will be more challenging. In the classroom setting, starting with the least possible number of assumptions may make proving early theorems long, difficult, and impractical. As Wylie states, "[i]n such cases, mathematical elegance must yield to pedagogical necessity, and one or more of the theorems in question should be included among the axioms," along with an admission that these results actually follow from a subset of the chosen axioms (a simpler approach is to just take the results as having proofs "beyond the scope of the course," at least at the beginning). Similar to the definition of "chaotic" mentioned above, David Hilbert (January 23, 1862 – February 14, 1943) introduced 21 axioms of Euclidean Geometry in his *Grundlagen der Geometrie* (1899), one of which was shown to be dependent on the other 20 (this was shown in E. H. Moore's "On the Projective Axioms of Geometry," *Transactions of the American Mathematical Society*, **3**, 142–158 (1902); a copy of this paper is on the American Mathematical Society website [accessed 9/19/2021]). An English edition of Hilbert's work (the title in English is "The Foundations of Geometry") is available from Project Gutenberg (accessed 9/19/2021). We are particularly interested in this part of Hilbert's work, since our text book attempts in Chapter 2 to present a similar modern study of classical Euclidean geometry.

Note. We deal with independence in a way similar to the technique of Section 1.4 involving showing consistence. Namely, we use models. To show that a particular axiom, Axiom A say, is independent of the other axioms in an axiomatic system, we consider the new axiomatic system consisting of the negation of Axiom A along with the other original axioms. If we show that this new axiomatic system is consistent (by giving a model for the new system), then Axiom A must be independent of the other axioms (or else the new axiomatic system could not be consistent). That is, for axiomatic system with axioms $A_1, A_2, \ldots, A_n$, we consider the new axiomatic system with axioms $A_1, A_2, \ldots, A_{k-1}, A_{k+1}, A_{k+2}, \ldots, A_n$ and the negation of axiom A_k. We give a model for the new axiom system which shows the consistence of the new axiomatic system and hence the independence of axiom A_k from the other axioms. Notice that this must be done for each $k \in \{1, 2, \ldots, n\}$ and so requires a total of n models.

Note. Recall that the consistent axiomatic system we dealt with in the previous section was:

A.1. If x_1 and x_2 are any two (distinct) x's, then there is at least one y belonging to both x_1 and x_2.

A.2. If x_1 and x_2 are any two x's, there is at most one y belonging to both x_1 and x_2.

A.3. If y_1 and y_2 are any two y's, there is at least one x belonging to both y_1 and y_2.

A.4. If y_1 is any y, there are at least three x's belonging to y_1.

A.5. If y_1 is any y, there is at least one x which does not belong to y_1.

A.6. There exists at least one y.

Consistence was given by the Model 2 in Figure 1.1 (with "belonging to" as described in Section 1.4). We now show independence by giving six appropriate models.

Note. Axiom A.1 is false when there are (distinct) x_1 and x_2 such that either (1) there is no y belonging to both x_1 and x_2, or (2) there is more than one y belonging to both x_1 and x_2. Consider the model of Figure 1.2 (with the same interpretation of "belonging to" as used in Figure 1.1).

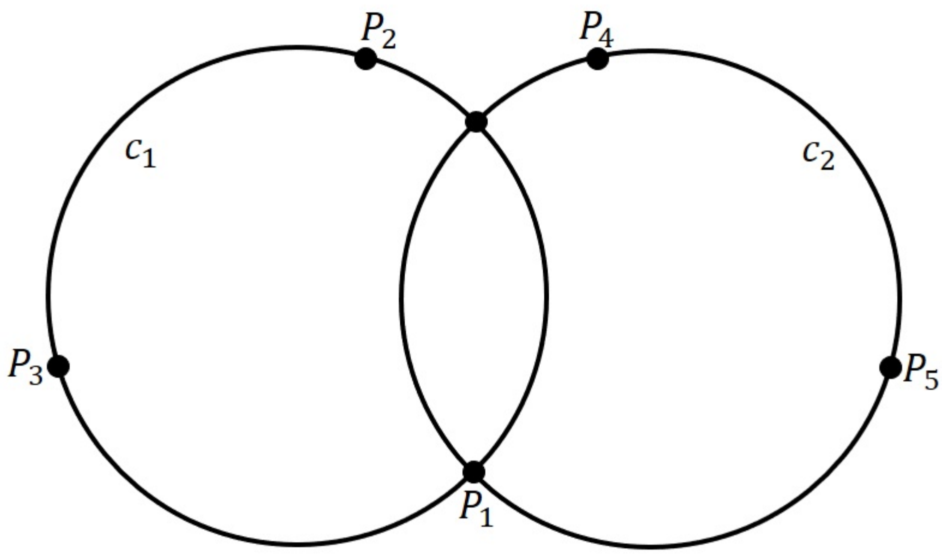

Figure 1.2

Here we have the set of x's is $X = \{P_1, P_2, P_3, P_4, P_5\}$ and the set of y's is $Y = \{c_1, c_2\}$. Notice that Axiom A.1 is false since there is no y belonging to both P_3 and P_4; that is, neither c_1 nor c_2 belongs to both P_3 and P_4. It is straightforward to verify the other axioms.

Note. Axiom A.2 is false when there are (distinct) x_1 and x_2 such that there is more than one y belonging to both x_1 and x_2. Consider the model of Figure 1.3 (with the same interpretation of "belonging to" as used in Figure 1.1).

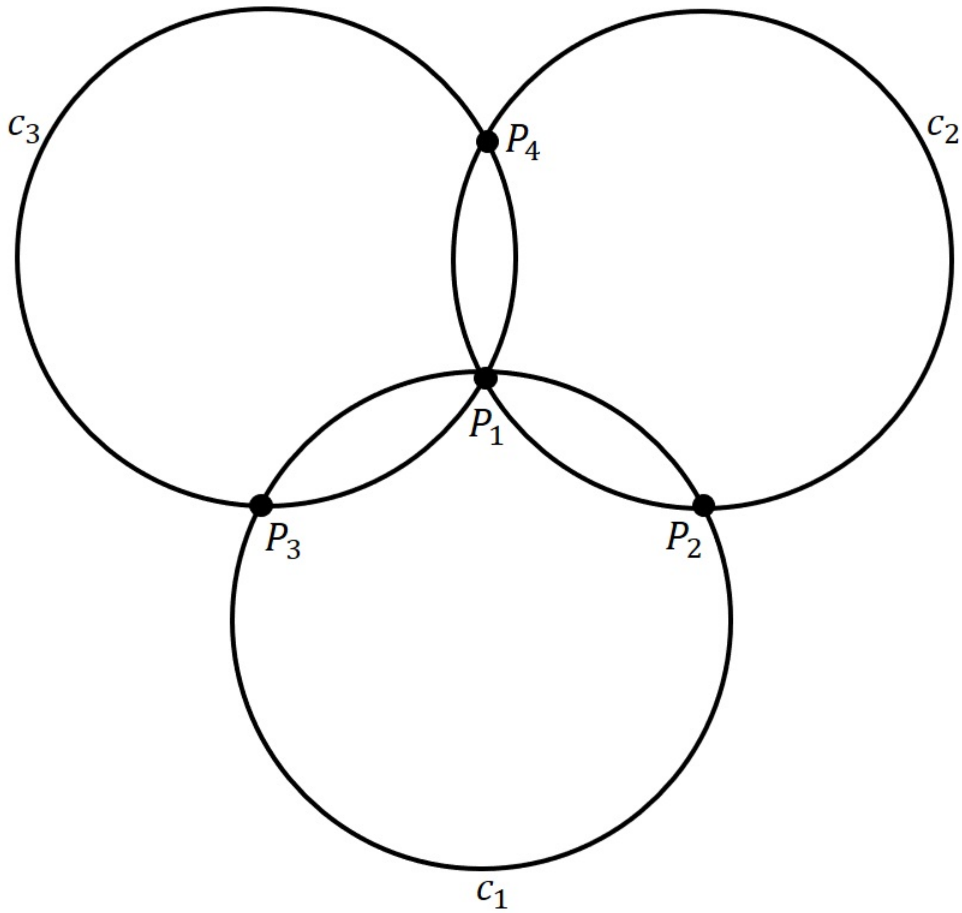

Figure 1.3

Here we have the set of x's is $X = \{P_1, P_2, P_3, P_4\}$ and the set of y's is $Y = \{c_1, c_2, c_3\}$. Notice that Axiom A.2 is false since there are two y belonging to both P_1 and P_2; that is, c_1 and c_2 both belong to P_1 and P_2 (there are similar problems with the pairs P_1, P_3 and P_1, P_4). It is straightforward to verify the other axioms.

Note. Axiom A.3 is false when there are y_1 and y_2 such that no x belongs to both y_1 and y_2. Here we can use ordinary plane Euclidean geometry as a model of the new axiomatic system. We let set X be the points in the Euclidean plane and let set Y be the set of lines in the Euclidean plane. We interpret "x belongs to y" to mean that point x lies on line y, and we interpret "y belongs to x" to mean that line y passes through point x. Axiom A.3 is false in this model, as we see when y_1 and y_2 are parallel lines (which, by the definition of parallel, share not points). The other axioms are easily seen to be satisfied.

Note. Axiom A.4 is false when there is some y such that less than three x's belong to y. We consider a somewhat different model for this new axiomatic system. Let $X = \{P, Q, R\}$ and $Y = \{\{P, Q\}, \{Q, R\}, \{R, P\}\}$. Then we say "$x$ belongs to y" if $x \in y$, and we say "y belongs to x if $x \in y$. Then Axiom A.4 is false since $y = \{P, Q\}$ has only two x's (namely P and Q) belonging to it (of course there is a similar situation with each $y \in Y$). The other axioms are easily seen to be satisfied. Wiley describes this model by considering the array

$$\begin{array}{ccc} P & Q & R \\ Q & R & P \end{array}$$

in which $X = \{P, Q, R\}$ and Y consists of the three columns. Then we say "x belongs to y" (and "y belongs to x") if and only if x lies in column y.

Note. Axiom A.5 is false when there is some y such that all x's belong to y. Consider Figure 1.4. Here we take $X = \{P_1, P_2, P_3\}$ and $Y = \{\ell\}$ and, as done in several cases above, interpret "belongs to" as either "lies on" or "passes through." Since all x's belong to ℓ, then Axiom A.5 is false. Axioms A.1, A.2, A.4, and A.6 are easily verified (since there are so few components in the model). Axiom A.3 states: "If y_1 and y_2 are any two y's, there is at least one x belonging to both y_1 and y_2." This is true *vacuously* since there are not two y's! That is, since the hypothesis is false then the conclusion is true.

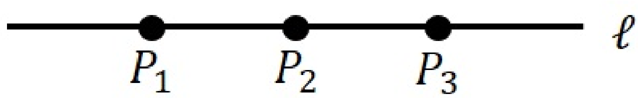

Figure 1.4

Note. Axiom A.6 is false when there is no y. So we consider the model $X = \{x\}$ and $Y = \varnothing$. Then Axiom A.6 is false. Each of the other Axioms A.1–A.5 are satisfied vacuously since each of these axioms hypothesize either multiple x's or at least on y.

Note. We have considered six axiomatic systems, each containing five of the original axioms and the negation of the sixth axiom, for the system of six axioms given in Section 1.4 We have absolute consistence for each of these six axiomatic systems by presenting a model for each (the consistence of the original six axioms is given in Section 1.4).