1.6. Completeness and Categoricalness

Note. In this section, we define a complete and a categorical axiomatic system. We discuss these ideas in the context of concrete models, and define isomorphic axiomatic systems. We conclude with a brief description of the work of Kurt Gödel on completeness.

Note. Within an axiomatic system, a meaningful statement involving the undefined terms and relations which is not a theorem has three possible statuses: (1) it may be a theorem and a proof of it deduced from the axioms exists, (2) it may be false and a proof of its negation exists, or (3) neither of these (it is an "undecidable" statement). In the third case, it must be that the axiomatic system is not strong enough to decide the statement. Perhaps more axioms can be added so that the statement can be proved or disproved (or the statement itself can be added as an axiom. . . or its negation can be added as an axiom). On the other hand, if every meaningful statement can either be proved or disproved then no new axioms can be added since any new axiom would be a meaningful statement and hence would be, in this case, either provable or disprovable; when provable it would already be an existing theorem and its addition as an axiom would introduce dependence, and when disprovable its addition as an axiom would make the axiomatic system inconsistent. These ideas inspire the next definition.

Definition. A set of axioms is *complete* if it is impossible to enlarge it be adding any other axiom which is consistent with, yet independent of, those already in the system.

Note. In a complete axiomatic system, every meaningful statement can either be proved or disproved; that is, there are no undecidable statements. It is hard to prove that a given set of axioms is complete. On approach is based on "categoricalness," which we discuss next. The idea of completeness is due to Kurt Gödel (April 28, 1906–January 14, 1978). In his 1929 doctoral dissertation at the University of Vienna he proved the completeness of a system called predicate calculus. He published this in "Die Vollständigkeit der Axiome des logischen Funktionenkalküls [The Completeness of the Axioms of the Logical Function Calculus]," *Monatshefte fr Mathematik* (in German), **37**(1), 349-360 (1930). We will discuss the work of Gödel in more detail at the end of this section.

Note. Consider (absolutely consistent) axiomatic system S with two specific models M_1 and M_2 each of which contain the same number of elements (or more generally, the cardinalities of the sets of elements for the two models are the same). So there is a one-to-one-correspondence (i.e., a bijection) between the elements of M_1 and the elements of M_2. Recall Model 1 of the axiomatic system from Section 1.4. Consistency:

	Amy	Berg	Cindy	Dunn	Ellen	Ford	Ginger
Math	B	B	B				
Sociology	B			B	B		
Physics	B					B	B
Astronomy		B		B		B	
Abstract Algebra		B			B		B
Engineering			B	B			B
History			B		B	B	

and Model 2 for the system:

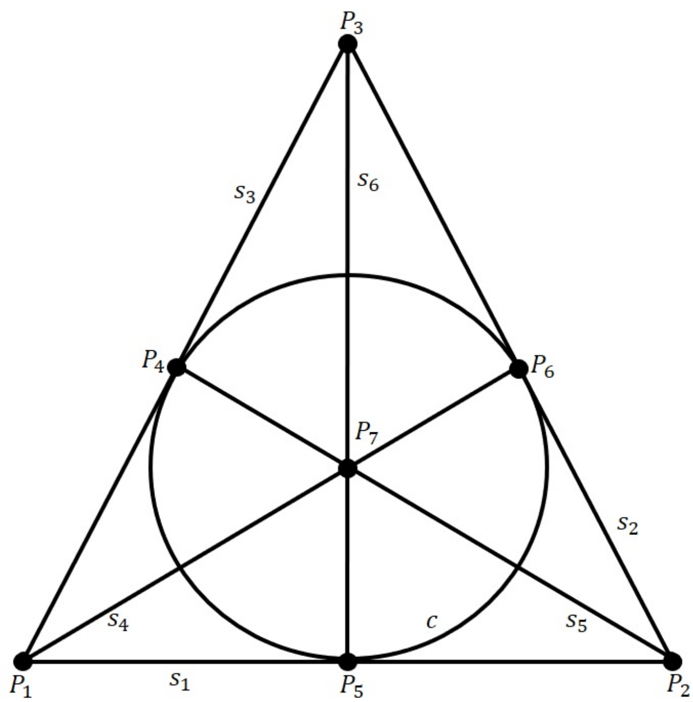

Figure 1.1. Model 2

Wylie describes a one-to-one-correspondence between these two models that *does not* result in the same collection of relations. Of more interest is a one-to-one-correspondence that *does* result in the same collection of relations. Consider the correspondences:

Model 1:	Amy	Berge	Cindy	Dunn	Ellen	Ford	Ginger
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
Model 2:	P_2	P_3	P_6	P_5	P_1	P_7	P_4

Math	Sociology	Physics	Astronomy	Abstract Algebra	Engineering	History
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
s_2	s_1	s_5	s_6	s_3	C	s_4

Under these correspondences, every relation in Model 1 corresponds to a relation in Model 2, and vice versa. This idea allows us to extend the idea of an isomorphism to the setting of axiomatic systems.

Definition. If there exists a one-to-one-correspondence between the elements of two axiomatic systems which preserves all relations existing in either system, then the correspondence is an *isomorphism* and the two systems are *isomorphic*.

Note. Two isomorphic axiomatic systems are structurally the same. The only difference in the names given to the elements. We can illustrate this by replacing the labels used in Figure 1.1 (which represents Model 2) with the labels from Model 1 (see Figure 1.5′ below). You are likely familiar with the idea of an isomorphism from other settings. An isomorphism is always a bijection that preserves structure. In an axiomatic system, the structure is the relations. In a graph, the structure is adjacency and an isomorphism between two graphs is a bijection between the vertex sets which preserves adjacency. In a group, the structure is the binary operation and an isomorphism between two groups is a bijection between the elements of the groups that preserves the binary operation. The structure of a vector space is the behavior of linear combinations of vectors. An isomorphism between two vector spaces (over the same scalar field) is a bijection between the sets of vectors which preserves linear combinations. I have online notes explaining these ideas for graphs in Section 1.2. Subgraphs, Isomorphic Graphs (for Introduction to Graph Theory [MATH 4347/5347]), for groups in Section I.3. Isomorphic Binary Structures (for Introduction to Modern Algebra [MATH 4127/5127]), and for vector spaces in Section 3.3. Coordinatization of Vectors (for Linear Algebra [MATH 2010]).

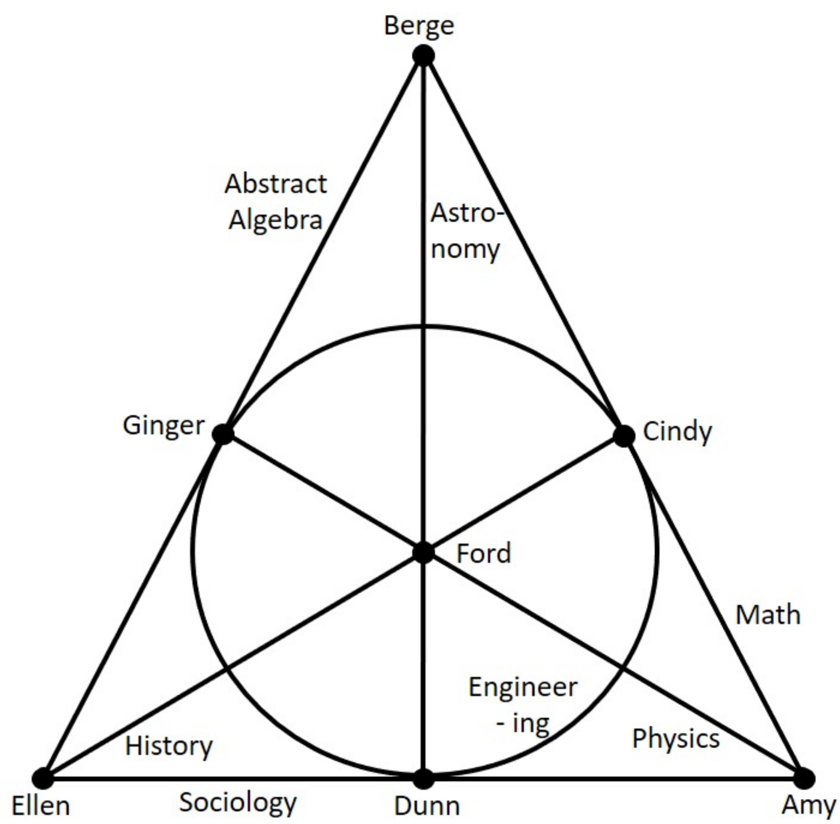

Figure 1.5′. A modified version of Wiley's Figure 1.5.

Definition. An axiomatic system is *categorical* if each of its models are isomorphic to ever other model.

Note. Informally, an axiomatic system is categorical if all of its models are "structurally the same." That is, there is "essentially" just one concrete representation of the system. Category theory is introduced in Modern Algebra 1 (MATH 5410) in Section I.7. Categories: Products, Coproducts, and Free Objects. If properties of a particular category of objects can be established, then one can show that another object has all those properties if they can show that the other object in in the particular category; this is the practical power of categories! In addition, categoricalness implies completeness, as we now argue.

Theorem 1.6.A. If an axiomatic system is categorical then it is complete.

Idea of the Proof. Consider an axiomatic system which is categorical. ASSUME that it is not complete. Since the system is not complete, then there is a statement σ which can neither be proved nor disproved (it is undecidable). This means that both σ and its negation are consistent with the axiomatic system. Consider two models of the given axiomatic system, on in which σ is true and one in which σ is false. Since the systems is categorical then, by definition, the two models are isomorphic. But isomorphic models have corresponding statements in the two models as both true or both false. But statement σ is true in one model and false in the other, so no such isomorphism can exist, a CONTRADICTION. So the assumption that a categorical axiomatic system is not complete is false, and hence every categorical axiomatic system must be complete, as claimed. ∎

Note. The axiomatic system of Section 1.4. Consistency is not categorical. Here we give Model 3 for the system. We consider a model with 13 x's (or "points") and 13 y's (or "lines"). We represent the points as P_i and the lines as ℓ_i, where $1 \leq i \leq 13$. In the following table, the columns represent the lines and the points they contain:

Model 3

ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}	ℓ_{13}
P_1	P_3	P_2	P_3	P_2	P_1	P_1	P_2	P_3	P_1	P_4	P_7	P_{10}
P_6	P_5	P_4	P_4	P_6	P_5	P_4	P_5	P_6	P_2	P_5	P_8	P_{11}
P_7	P_8	P_9	P_7	P_8	P_9	P_8	P_7	P_9	P_3	P_6	P_9	P_{12}
P_{10}	P_{10}	P_{10}	P_{11}	P_{11}	P_{11}	P_{12}	P_{12}	P_{12}	P_{13}	P_{13}	P_{13}	P_{13}

Since this contains 13 x's and 13 y's, then this model cannot be isomorphic to Model 1 or Model 2 (both of which contain 7 x's and 7 y's. Notice in Model 3 that each y (i.e., line) contains exactly 4 x's (i.e., points), and each x contains exactly 4 y's.

Note. Wiley states on page 27 that "...the system is not categorical, and therefore not complete...." However, this does not follow from Theorem 1.6.A. It is the contrapositive of the converse of Theorem 1.6.A, and so need not hold. In fact, according to the Wikipedia page on Axiomatic Systems (accessed 10/20/2021; not exactly an academic reference), "Completeness does not ensure the categoriality (categoricity) of a system, since two models can differ in properties that cannot be expressed by the semantics of the system." Though not guaranteed, we do have in the case of the axiomatic system of Section 1.4 that it is both not categorical (as just demonstrated) and not complete. Consider the statement σ: "If y_1 is a y, there are at most three x's which belong to y_1." This statement is true in Models 1 and 2 and false in Model 3. If we add σ as an axiom to $A.1$, $A.2$, $A.3$, $A.4$, $A.5$, and $A.6$ then we get an axiomatic system which we claim is categorical (two isomorphic models being Models 1 and 2), and hence is complete by Theorem 1.6.A.

Note. If we are dealing with a noncategorical axiomatic system, then any theorems we prove within the system are valid for every model for the system. One example is the axiomatic system of Section 1.4 with axioms $A.1$ through $A.6$. Another example is called "neutral geometry" which is a collection of results valid in both Euclidean geometry and non-Euclidean geometry. On the other hand, if we are interested in one particular model (such as Euclidean geometry), then we would want a categorical set of axioms that admit that particular model (and that is the "only model" for the system, up to isomorphism).

Note. As mentioned above, Kurt Gödel introduced the idea of completeness in an axiomatic system. We now elaborate on some more details related to completeness. This information is from R. Goldstein's *Incompleteness: The Proof and Paradox of Kurt Gödel*, W. W. Norton, Great Discoveries Series (2005). The rules of manipulation in a formal axiomatic system are of three sorts: (1) the rules that specify what the symbols are (the "alphabet" of the system), (2) the rules that specify how the symbols can be put together to make "well-formed formulas" (or 'WFF's) which include the claims of the system (the lemmas, corollaries, and theorems), and (3) the rules of inference that specify which WFFs can be derived from other WFFs [Goldstein, page 86].

Kurt Gödel (April 28, 1906–January 14, 1978), image from MacTutor History of Mathematics Archive biography of Gödel.

An axiomatic system is *complete* if a truth value can be put on every WFF. What this means is that every meaningful statement can either be proven to be true or false. (In geometry, examples of WFFs are: "Two lines, parallel to a third, are parallel to each other" [true in Euclidean geometry], and "The sum of the measures of the angles of a triangle is less than 180°" [false in Euclidean geometry]. An example of a statement which involves the objects of geometry, but is not a WFF [since it is meaningless] is: "All points are parallel.") As mentioned above, Gödel proved the completeness of a system called predicate calculus. However, he is better known from his two main results on incompleteness:

Gödel's First Incompleteness Theorem. There are provably unprovable but nonetheless true propositions in any formal system that contains elementary arithmetic, assuming that system to be consistent.

Gödel's Second Incompleteness Theorem. The consistency of a formal system that contains arithmetic can't be formally proved with that system [Goldstein, page 183].

What Gödel *has* shown is that there are meaningful statements in axiomatic systems (which include arithmetic) which can neither be proved to be true nor proved to be false. Such statements are said to be *undecidable*. A specific example of this is the *Continuum Hypothesis* which addresses the existence of a set of real numbers S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$ (where $|S|$ is the cardinality of set S). This idea of undecidability was disturbing to a number of pure mathematicians of the time. After all, these results, if not contradicting the work of Frege and Russell (who sought to put all of mathematics on an axiomatic foundation using set theory), is in direct contrast to the *spirit* of these works. However, other mathematicians took the results in stride with the attitude that mathematics will continue to go forward and undecidable propositions are now just part of the terrain. For more information on these ideas (with some comments about the work of Frege, Russell, and Hilbert), see my online presentation on Introduction to Math Philosophy and Meaning (prepared for use in various junior and senior level pure math classes).

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