1.7. Finite Geometries

Note. In this section, we state the six axioms of finite projective geometry. In fact, this is the same axiomatic system as we encountered in Section 1.4. Consistency. We prove several theorems in this system and introduce the idea of duality in this system.

Note. We consider again the axiomatic system of Section 1.4. Consistency. We have considered this system simply as an example of an axiomatic system with which we illustrated the ideas of consistency, independence, completeness, and categoricalness. We now consider the axiomatic systems in defining finite projective geometries. To do so, we reword the axioms by replacing x with "point" and y with "line." We also replace "belongs to" (which replace with the statements that a point "lies on" or "belongs to" a line, or that a line "passes through" or "contains" a point).

Definition. A *finite projective geometry* consists of the undefined terms "points" and "lines" which satisfy the following axioms.

A.1. If P_1 and P_2 are any two points, there is at least one line containing both P_1 and P_2.

A.2. If P_1 and P_2 are any two points, there is at most one line containing both P_1 and P_2.

A.3. If ℓ_1 and ℓ_2 are any two lines, then there is at least one point which lies on both ℓ_1 and ℓ_2.

A.4. Every line contains at least three points.

A.5. If ℓ is any line, then there is at least one point which does not lie on ℓ.

A.6. There exists at least one line.

Note. We know from Section 1.4. Consistency that these axioms are (absolutely) consistent. We know by the long argument of Section 1.5. Independence that these axioms are independent. We know from the results of Section 1.6. Completeness and Categoricalness that this system is neither complete (as shown by the undecidable statement σ "If ℓ_1 is a line, there are at most three points which belong to ℓ_1.") nor categorical (as shown by the two nonisomorphic models Model 2 and Model 3 of Section 1.6). Wiley points out that A.3, "if ℓ_1 and ℓ_2 are any two lines, then there is at least one point which lies on both ℓ_1 and ℓ_2," is not a "common sense" result and that it is, in fact, false in Euclidean geometry where nonintersecting (i.e., parallel) lines exist.

Note. We now prove several theorems in the axiomatic system of finite projective geometry. Wiley observes that we cannot rely on our intuitive ideas about lines and points: "A point is not a small dot, nor even the limit of smaller and smaller dots made with sharper and sharper pencils. Not is a line an indefinitely long, indefinitely thin mark made with an arbitrarily sharp pencil. Mathematically speaking, *a point is any object that has the properties the axioms say a point should have*, and *a line is any object that has the properties the axioms say a line should have*" (page 29; his emphasis).

Theorem 1.7.1. There exists at least one point.

Note. The next result is really just a restatement of axiom A.2, but we give a brief proof anyway.

Theorem 1.7.2. If ℓ_1 and ℓ_2 are any two lines, there is at most one point which lies on both ℓ_1 and ℓ_2.

Theorem 1.7.3. Two points determine exactly one line.

Theorem 1.7.4. Two lines have exactly one point in common.

Note. The proofs of the remaining results of this section are a bit more involved.

Theorem 1.7.5. If P is any point, there is at least one line which does not pass through P.

Theorem 1.7.6. Every point lies on at least three lines.

Note. Notice the following pairing of axioms and theorems from this section:

A.1. If P_1 and P_2 are any two points, there is at least one line containing both P_1 and P_2.	A.3. If ℓ_1 and ℓ_2 are any two lines, then there is at least one point which lies on both ℓ_1 and ℓ_2.
A.2. If P_1 and P_2 are any two points, there is at most one line containing both P_1 and P_2.	Thm 2. If ℓ_1 and ℓ_2 are any two lines, there is at most one point which lies on both ℓ_1 and ℓ_2.
A.4. Every line contains at least three points.	Thm 6. Every point lies on at least three lines.
A.5. If ℓ is any line, then there is at least one point which does not lie on ℓ.	Thm 5. If P is any point, then there is at least one line which does not pass through P.
A.6. There exists at least one line.	Thm 1. There exists at least one point.

Each statement in the first column has exactly the same structure as the corresponding statement in the second column, except that the terms "point" and "line" have been interchanged (and the terms "lies on," "passes through," and "contains" are introduced as appropriate). Since each of axioms A.1 to A.6 are included in this table, then for any statement proved within the axiomatic system about a relationship between lines and points, there is a corresponding statement which will immediately follow concerning a relationship between points and lines (respectively).

Definition. Two statements in the axiomatic system of finite projective geometry which differ only in the interchange of the words "point" and "line" (and the associated changes in "lies on," "passes through," and "contains") are *duals* of each other. The fact that the dual of any theorem is also a theorem is the *principle of duality* of finite projective geometry.

Note. Next, we focus on results concerning the numbers of points and lines (and their enumerative relationships) in any finite projective geometry.

Theorem 1.7.7. If there exists one line which contains exactly n points, then every line contains exactly n points.

Theorem 1.7.8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Theorem 1.7.9. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ points.

Theorem 1.7.10. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ lines.

Note. Notice that Theorems 1.7.7 to 1.7.10 do not insure the existence of any particular structures, but instead are a conditional result. From these results, we see that the number n of points on a line determines several properties of a finite projective geometry. O. Veblen and W. H. Bussey published "Finite Projective Geometries," *Transactions of the American Mathematical Society*, **7**, 241–259 (1906); a copy is available online on the AMS website. In this, they showed that for every value of n such that $n - 1$ is a power of a prime, a finite projective geometry in which each line contains n points exists. R. H. Bruck and H. J. Ryser in "The Nonexistence of Certain Finite Projective Planes," *Canadian Journal of Mathematics*, **1**, 88–93 (1949) proved that if a projective plane with n points on each line exists where $n - 1 \equiv 1$ or $2 \pmod 4$, then $n - 1$ is the sum of two squares. This is sometimes called the "Bruck-Ryser-Chowla Theorem." Since $7 - 1 \equiv 2 \pmod 4$ and $7 - 1 = 6$ is not the sum of two squares, then no finite projective geometry exists with 7 points on each line.

Note. Wylie concludes this section with the comment "...it is not known whether there exists a finite projective geometry in which every line contains exactly 11 points." See page 36. This question has been resolved since the publication of the book. Using a computer search, C. W. H. Lam, L. H. Thiel, and S. Swiercz showed that such a system does not exist in "The Non-Existence of Finite Projective Planes of Order 10," *Canadian Journal of Mathematics*, **XLI**, 1117–1123 (1989). It is common to replace the parameter n of Wylie with the parameter $n - 1$, so that a finite projective plane of "order 10" actually has 11 points per line (if you read about the results mentioned in the previous note, then you are likely to find the use of the term "order"). The story of the development of this computer search is recounted in Lam's "The Search for a Finite Projective Plane of Order 10," *The American Mathematical Monthly*, **98**(4), 305–318 (April 1991). This is available online on JSTOR. Clement Lam was a Ph.D. student of Herbert J. Ryser, whose 1949 paper is mentioned above. According to the Wolfram MathWorld webpage on Projective Planes, "The status of the order 12 projective plane remains open." Notice that this deals with a system where the lines all have 13 points. These websites were accessed 10/23/2021.

Revised: 10/23/2021