2.2. A Brief Critique of Euclid

Note. In this section we discuss some weaknesses of Euclid's approach to geometry. We have already mentioned the futility of trying to define every term. We will also address Euclid's failure to deal with continuity appropriately and give a "proof" that every triangle is isosceles by taking advantage of a Euclid's weakness in dealing with order relations (and "betweenness").

Note. We mentioned in Section 1.3. Axiomatic Systems that an axiomatic system includes undefined terms. Euclid does not take this approach and he has the following definitions of "point," "straight line" (which is distinguished from a "line"), and "plane angle."

Definition. A *point* is that which has no part.

Definition. A *straight line* is a line which lies evenly with the points on itself.

Definition. A *plane angle* is the inclination to one another of two straight lines in a plane which meet one another and do not lie in a straight line.

Of course this raises as many questions as it answers, since we now focus on the terms "part," "lies evenly," and "inclination."

Note. The next concern deals with continuity. Many of the results in the *Elements* are inspired by compass and straight-edge constructions. In Book I Proposition I Euclid presents the construction of an equilateral triangle in which additional, unstated assumptions are needed. As in Figure 2.1(b), he starts with a line segment, uses the compass to draw an arc of a circle of radius equal to the length of the segment and centered at one end of the segment, and then uses the compass to draw an arc of another circle of the same radius and centered at the other end of the segment. Of course, this results in a point that is equidistant from both endpoints of the segment and which can be used to construct the desired isosceles triangle. This is shown in Figure 2.1(a) for the point "above" the line segment (we know there is also such a point "below" the line segment). The continuity concern deals with the intersections of the arcs of the circles. What if the lines have "holes" in them or if the points on the arcs are distributed in such a way ("like beads on a string," as Wiley says on page 40) that the arcs can pass through each other without intersecting? This is illustrated somewhat in Figure 2.1(b).

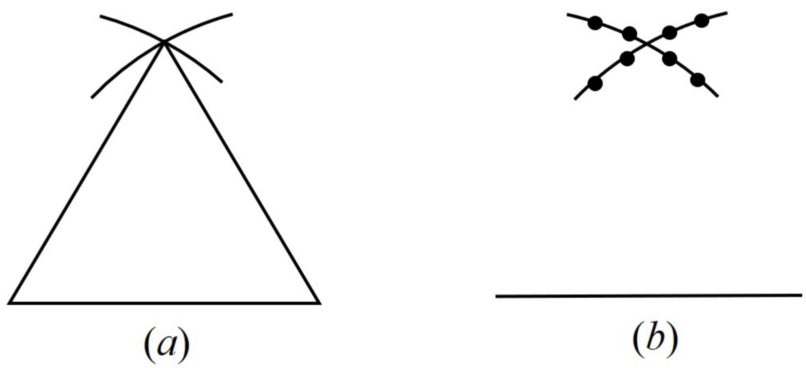

Figure 2.1

Note. It might sound odd to worry about these details which certainly violate our intuitive ideas of circles. In fact, this plays out to a resolution in the 19th century. Wendell Strong in "Is Continuity of Space Necessary to Euclid's Geometry?" *Bulletin of the American Mathematical Society*, **4**(9), 448–448 (June 1898) (a copy can be downloaded from projecteuclid.org) discusses what he calls *quadratic space*. This space consists of all points in the Cartesian plane which have quadratic coordinates (a real number is *quadratic* if it can be obtained from the integers by a finite number of rational operations and extractions of square roots). The quadratic space is everywhere discontinuous, yet any construction that can be performed with a compass and a straightedge can be performed in this space! So in response to Strong's question in the title of his paper, "No!" However, two circles with quadratic centers and quadratic radii which intersect in the continuous Cartesian plane intersect at quadratic points. By restricting our attention to intersections of lines and circles (which are themselves constructible from existing [constructible] points and distances), continuity is not needed! This certainly was not known to Euclid and was not resolved until the study of field theory in modern algebra arose. Though not actually developed with these problems in mind, the area of field theory in algebra ultimately is the tool allowing us to classify Strong's quadratic numbers. This is covered in our Introduction to Modern Algebra 2 (MATH 4137/5137) in Section VI.32. Geometric Constructions and our Modern Algebra 2 (MATH 5420) in Section V.1.Appendix. Ruler and Compass Constructions. The rational numbers are constructible as seen in Section 1.2. Similar Figures (see Figure 1.6). It can be shown that the set of real constructible numbers C forms a subfield of the field of real numbers (see Corollary 32.5 in the Introduction to Modern Algebra 2 notes) and, in particular, the field of constructible real numbers C consists precisely of all real numbers that we can obtain from $\mathbb{Q}$ by taking square roots of positive numbers a finite number of times and applying a finite number of field operations. From the Introduction to Modern Algebra 2 we have:

> **Theorem 32.6.** The field of constructible real numbers consists precisely of all real numbers that we can obtain from Q by taking square roots of positive numbers a finite number of times and applying a finite number of field operations

See also Proposition V.1.16 in Section V.1.Appendix. Ruler and Compass Constructions of Modern Algebra 2. Additional details on constructions can be found in my video "Compass Straightedge Constructions" on YouTube. This study of constructible numbers ultimately grows out of the study of the three classical compass and straightedge constructibility problems (Doubling the Cube, Squaring the Circle, and Trisecting an Angle); these problems also serve as the original inspiration for the study of the conic sections. For more detail on the history, see my online notes for "Introduction to Modern Geometry (History)" (MATH 4157/5157) on Section 1.8. Three Famous Problems of Greek Geometry and Chapter 3. Conic Sections.

Note. Interestingly, the complexities of a continuum were first rigorously explored in the work of Richard Dedekind (October 6, 1831–February 12, 1916). The real line is a continuum due to the Axiom of Completeness. This is traditionally dealt with in terms sets of real numbers with upper bounds. Concisely, the Axiom of Completeness states that every set of real numbers with an upper bound has a *least* upper bound. For details, see my online notes for Analysis 1 (MATH 4217/5217) on Section 1.3. The Completeness Axiom. An alternative approach is the use of "Dedekind cuts." This idea was introduced in Dedekind's 1858 work (first published in 1872) "Continuity and Irrational Numbers" (a copy can be found online at

Project Gutenberg). These ideas are informally explained in the setting of an airplane taking off in my online Calculus 1 (MATH 1910) notes on Appendix A.6. Theory of the Real Numbers. I also have a video of this material at A.6 Video (of length (31:05)). On a personal note, I find it amazing that the idea of a line as a continuum is as old as any other in geometry, but it was not until 1858 that an appropriate axiomatic system was introduced to deal with this idea!

Note. Another critique of Euclid relates to the concept of "betweenness." That is, the order in which points appear on a line is largely left unaddressed; drawings are used and "...it was sometimes for [Euclid] to establish with certainty the location of one point with respect to others in a given discussion." See Wylie's pages 40 and 41. To illustrate this, we consider the following absurd result. This is a widely circulated idea; a Google search of "all triangles are isosceles" gives about 5000 sites (as of 10/26/2021).

"Theorem." Every triangle is isosceles.

"Proof." Let $\triangle ABC$ by an arbitrary triangle. Let the bisector of $\angle BAC$, introduce the perpendicular bisector of side BC, and let O be their point of intersection, as shown in Figure 2.2 below. Let points A', B', C' be, respectively, the points of intersection of the perpendiculars from O to the sides BC, CA, and AB. Notice by our construction that A' is also the midpoint of side BC.

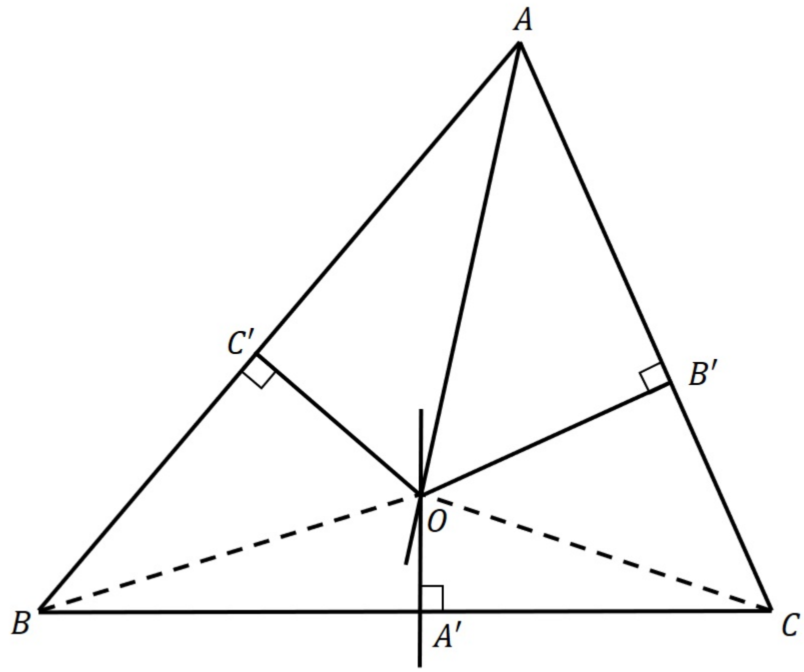

Figure 2.2

We now present our argument in the two column style that you may have used in high school geometry (we also use a notation for angles and triangles from high school).

1. $A'O = A'O$.	**1.** Identity.
2. $BA' = A'C$.	**2.** By construction.
3. $\angle OA'B = \angle OA'C$.	**3.** Both are right angles.
4. $\triangle OA'B \cong \triangle OA'C$.	**4.** Side, Angle, Side.
5. $OB = OC$.	**5.** Corresponding parts of congruent $\triangle$'s.
6. $AO = AO$.	**6.** Identity.
7. $\angle C'AO = \angle B'AO$.	**7.** AO bisects $\angle BAC$.
8. $\angle AC'O = \angle AB'O$.	**8.** Both are right angles.
9. $\triangle AC'O \cong \triangle AB'O$.	**9.** Angle, Side, Angle.
10. $AC' = AB'$.	**10.** Corresponding parts of congruent $\triangle$'s.

11. $OC' = OB'$.	**11.** Corresponding parts of congruent $\triangle$'s.
12. $\angle OC'B = \angle OB'C$.	**12.** Both are right angles.
13. $OB = OC$.	**13.** By Step 5.
14. $\triangle OC'B \cong \triangle OB'C$.	**14.** Right angle, hypotenuse, leg.
15. $C'B = B'C$.	**15.** Corresponding parts of congruent $\triangle$'s.
16. $AB = AC' + C'B$.	**16.** C' is between A and B.
17. $AB = AB' + B'C$.	**17.** Steps 10 and 15.
18. $AC = AB' + B'C$.	**18.** B' is between A and C.
19. $AB = AC$.	**19.** Steps 17 and 18.

Therefore $\triangle ABC$ is an isosceles triangle. Since $\triangle ABC$ is an arbitrary triangle, then all triangles are isosceles, as claimed. *Q.E.D.* (NOT!)

Note. The problem in the previous "proof" is one of betweenness. The point O of intersection of the bisector of $\angle BAC$ and the perpendicular bisector of BC in fact lies outside of $\triangle ABC$ (as is to be shown in Exercise 2.11.10), so that one of the points B' and C' will lie *between* two vertices of the triangle while the other will not (it will lie on one of the sides, AC or AB respectively, extended beyond the vertices of $\triangle ABC$). We can see in Figure 2.2 that $\angle BAO$ appears a little smaller than $\angle CAO$, so that $\angle BAC$ is not actually bisected by AO.

Note. Since we desire an axiomatic system for Euclidean geometry which will leave the terms "point" and "line" as undefined (their meaning being given by the axioms and the theorems which follow from the axioms), we cannot appeal to figures and drawings! Wylie makes the following comment about the situation (see page 42):

> "As a matter of fact, from the axioms of Euclid it is impossible to determine whether the point O is inside or outside the triangle. Worse yet, it is impossible using Euclid's axioms, even to give a satisfactory definition of the inside and outside of a triangle."

Additional critiques can be aimed at Euclid's treatment of distances, the measures of angles, and the idea of congruence which is thought of in terms of superimposing one object (angle or triangle, for example) on another and requires some concept of transformations. In the remainder of this chapter, we address these problems and present an axiomatic system for Euclidean geometry.

Revised: 10/27/2021