2.3. The Postulate of Connection

Note. In this section we give the axiomatic system for Euclidean geometry. Instead of the term "axiom," we use the term "postulate." The undefined terms we use are "point," "line," and "plane." The postulates give these terms the meanings they have. Of course we choose the postulates in such a way as to give them the meanings which we already intuitively have about them. Wylie emphasizes (on page 46) that "...we must remember that their meaning comes solely from the postulates and not from any reference to counterparts in the external world."

Note. We will ultimately state 21 postulates. We start in this section with 7 postulates of "incidence" or "connection." The relations between the undefined terms (point, line, and plane) will include "contains," "lies in," "intersects," and "determines." We start with some definitions from set theory. For a more thorough exploration of set theory, see my online notes for Introduction to Set Theory. We number the postulates sequentially in this chapter, without prefacing the numbers with section numbers, but continue to number other things using the section number.

Definition 2.3.1. A set, S_1, is a *subset* of a set S if every member of S_1 is a member of S. A subset S_1 is a *proper subset* of S if there is at least one member of S which is not a member of S_1.

Definition 2.3.2. The *union* of the sets S_1 and S_2 is the set of all elements which are members of at least one of the sets S_1 and S_2.

Definition 2.3.3. The *intersection* of the sets S_1 and S_2 is the set of all elements which are simultaneously members of S_1 and members of S_2.

Definition 2.3.A. The *null set* or *empty set*, denoted $\varnothing$, is the set having no members.

Note. Though we leave the term "point" undefined, we are using the terminology of set theory to define "line" (in Postulate 1) and "plane" (in Postulate 3). We could argue that the undefined terms are "set," "element," and "point."

Postulate 1. Every line is a set of points and contains at least two points.

Postulate 2. If P and Q are two points, there is one and only one line which contains them both. We denote this line as $\overleftrightarrow{PQ}$ or $\overleftrightarrow{QP}$.

Definition 2.3.4. The points of a set are *collinear* if and only if there is a line which contains them all. Three or more points which do not lie on the same line are *noncollinear*.

Postulate 3. A plane is a set of points and contains at least three noncollinear points.

Postulate 4. If P, Q, and R are three noncollinear points, there is one and only one plane which contains them all.

Postulate 5. If two points of a line lie in a plane, then every point of the line lies in that plane.

Postulate 6. If two (distinct) planes have a point in common, their intersection is a line.

Definition 2.3.5. The points of a set are said to be *coplanar* if there is a plane which contains them all.

Definition 2.3.6. The set of all points is called *space*.

Note. The next postulate, for the first time, actually guarantees the existence of something.

Postulate 7. Space contains at least four points which are noncollinear and noncoplanar.

Note. We see from Postulates 2 and 7 that there exists at least one line. Postulates 4 and 7 imply that there exists at least one plane. In Euclidean geometry, we expect infinitely many points, lines, and planes. However, this is not implied by Postulates 1 through 7 alone. In fact, in Exercise 2.3.8 a system with 8 points, 28 lines, and 14 planes is given which satisfies Postulates 1 through 7.

Note. We now state our first 7 theorems. Proofs are to be given in Exercises 2.3.1 to 2.3.7, respectively. However, we give proofs of Theorems 2.3.1 and 2.3.4 for the sake of illustration.

Theorem 2.3.1. If two lines intersect, their intersection is a point.

Theorem 2.3.2. If a line intersects a plane which does not contain the line, the intersection is a point.

Theorem 2.3.3. If two planes intersect, their intersection is a line.

Theorem 2.3.4. A line and a point not on the line determine a unique plane.

Theorem 2.3.5. Two (distinct) intersecting lines determine a unique plane

Theorem 2.3.6. Space contains at least six lines.

Theorem 2.3.7. Space contains at least four planes.

Revised: 10/28/2021