2.4. The Measurements of Distance

Note. In this section we state four more postulates. The goal of this section is to introduce a coordinate system to any line in Euclidean geometry. In this way, every line becomes a "number line." We just need a pair of points to set up the unit distance and we start there.

Postulate 8. If A and A' are distinct points, there exists a correspondence which associates with each (unordered) pair of points in space a unique real number, such that the number assigned to a pair of points:

(1) is 0 if the points of the pair are the same,

(2) is positive if the points of the pair are distinct, and

(3) is 1 if the points of the pair at the given points, A and A'.

Definition. The set $\alpha = \{A, A'\}$ of two given points in Postulate 8 is the *unit pair*. The real number which corresponds a given pair of points in Postulate 8 is the *measure of the distance between the points* relative to the unit pair $\{A, A'\}$. We denote this measure of distance between points P and Q as $m_\alpha(P, Q)$ of $m_\alpha(Q, P)$.

Note. The next postulate allows us to convert from one measure of distance (based on unit pair $\{A, A'\}$) to another measure of distance (based on unit pair $\{B, B'\}$.

Postulate 9. If $\alpha = \{A, A'\}$ and $\beta = \{B, B'\}$ are two pairs of distinct points, then for all pairs of points P, Q we have $m_\alpha(P, Q) = m_\alpha(B, B')m_\beta(P, Q)$.

Note. The next result gives a sort-of invariance of sums of distances. We will use this in the next section when we address "betweenness." The result following it (Theorem 2.4.2) is similar, but instead of dealing with sums of distances it deals with ratios of distances; its proof is to be given in Exercise 2.4.1.

Theorem 2.4.1. If P, Q, and R are points such that for some unit pair, $\alpha = \{A, A'\}$, $m_\alpha(P, Q)+m_\alpha(Q, R) = m_\alpha(P, R)$ then for any other unit pair $\beta = \{B, B'\}$, we have $m_\beta(P, Q) + m_\beta(Q, R) = m_\beta(P, R)$.

Theorem 2.4.2. If P, Q, R, and S are points such that for some unit pair, $\alpha = \{A, A'\}$, we have $\dfrac{m_\alpha(P, Q)}{m_\alpha(R, S)} = k$ then for any other unit pair $\beta = \{B, B'\}$ we have $\dfrac{m_\beta(P, Q)}{m_\beta(R, S)} = k$.

Note. Notice that, so far, the postulates give no indication as to *how to create* the assignment of real numbers to pairs of points which will satisfy all of the postulates. Of course, we are trying to axiomatize the idea of using a ruler to measure the distance between points. The next postulate allows us to find find point(s) on a line of distance 1 from a given point (this then gives us a new unit pair of points which are on the line).

Postulate 10. If P is an arbitrary point on an arbitrary line, ℓ, and if $\alpha = \{A, A'\}$ is any unit pair, then there exists at least one point, Q, on ℓ such that $m_\alpha(P, Q) = 1$.

Note. Our last postulate of distance is so suggestive of the idea of using a ruler to measure distance that it is called "The Ruler Postulate."

Postulate 11. The Ruler Postulate. Let P be an arbitrary point on an arbitrary line, ℓ, and let $\alpha = \{A, A'\}$ be any unit pair. Then if Q is any point on ℓ such that $m_\alpha(P, Q) = 1$, there is a unique one-to-one correspondence between the set of all real numbers and the set of all points on ℓ such that:

(1) the number 0 corresponds to the point P,

(2) the number 1 corresponds to the point A, and

(3) the measure of the distance between any points R and S on ℓ relative to the unit pair $\alpha = \{A, A'\}$ is equal to the absolute value of the difference of the numbers which correspond to R and S.

Note. Wylie states on pages 54 and 55:

> "Postulate 11 is arithmetic rather then geometric in character and has no counterpart among the axioms employed in the traditional presentation of [E]uclidean geometry. Some mathematicians, inclining to the view that algebra and geometry should be kept apart until they are fin-

> -ally fused in analytic geometry, find Postulate 11 out of place and distasteful. It is an extremely powerful postulate, however, since it brings to bear on the problems of elementary geometry a large body of results from arithmetic."

With the introduction of a one-to-one correspondence between lines in Euclidean geometry and the real numbers, we have the equipment to deal with the betweenness and continuity issues which were raised in Section 2.2. A Brief Critique of Euclid (especially with the introduction of a coordinate system as given in the next definition). Of course this one-to-one correspondence now eliminates the possibility of a finite geometry in our axiomatic system! The properties of the real number are the topic of our class Analysis 1 (MATH 4217/5217); see my online notes for Analysis 1, especially the notes for "Chapter 1. The Real Number System."

Definition 2.4.1. A *coordinate system* on an arbitrary line, ℓ, is a one-to-one correspondence between the set of all points on ℓ and the set of all real numbers such that if P and Q are the points which correspond to 0 and 1, respectively, then the *measure of the distance* between any points, R and S, on ℓ relative to the unit pair $\{P, Q\}$ is equal to the absolute value of the difference of the numbers which correspond to R and S. The point which corresponds to the number 0 is the *origin*, and the point which corresponds to the number 1 is a *unit point*. The number which is assigned to any point by the coordinate system is the *coordinate* of that point.

Note. There are three compass and straightedge constructions that are impossible within the postulates and axioms of Euclid's *Elements*; see my online notes for Introduction to Modern Geometry—History (MATH 4157/5157) on Section 1.8. Three Famous Problems of Greek Geometry. These problems include doubling the cube and squaring the circle. These two problems could be solved if the real numbers $\sqrt[3]{2}$ and $\sqrt{\pi}$ could be constructed using a compass and straightedge. However, these numbers are not constructible as is shown in Introduction to Modern Algebra (MATH 4127/5127) in Section VI.32. Geometric Constructions and in Modern Algebra 2 (MATH 5420) in Section V.1 Appendix. Ruler and Compass Constructions (see Corollaries V.1.18 and V.1.19). However, in our setting we have the existence of a line segment of any given positive length by the Ruler Postulate (Postulate 11). Of course the line segments are not constructed with a compass and straightedge here, but instead by using the Ruler Postulate.

Note. Of course we can now use coordinates to deal with the (formerly ambiguous) idea of "betweenness." In the next section we turn our attention to this idea and order relations.

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