2.5. Order Relations

Note. In this section we use the ordering of the real numbers $\mathbb{R}$ to put an order relation on any line. For the formal definition of the ordering on $\mathbb{R}$, see my online notes for Analysis 1 (MATH 4217/5217) on Section 1.2. Properties of the Real Numbers as an Ordered Field; notice that the existence of an ordering (in terms of the existence of a positive subset) is an axiom of the real numbers. In what follows, when a unit pair $\alpha\{A, A'\}$ is given (or understood), we denote the distance between points A and B, $m_\alpha(A, B)$, simply as AB.

Definition 2.5.1. A point B is said to be *between* the points A and C if and only if:

(1) A, B, and C are distinct collinear points,

(2) $AB + BC = AC$.

Note. Notice that if $AB + BC = AC$ with respect to one coordinate system, then by Theorem 2.4.1 this equality holds with respect to any coordinate system. So the idea of "between" is independent of the coordinate system used. In the next theorem we see that betweenness on the real line (in terms of coordinates) translates directly into betweenness on line ℓ.

Theorem 2.5.1. Let A, B, and C be three points on line ℓ and let x, y, and z be, respectively. The coordinates of these points in a coordinate system on ℓ. Then B is between A and C if and only if y is between x and z.

Corollary 2.5.1. Of three collinear points, one and only one is between the other two.

Definition 2.5.2. If A and B are distinct points, the set consisting of A, B, and all points which are between A and B is called a *segment*. This segment is denoted $\overline{AB}$ or $\overline{BA}$. Points A and B are the *endpoints* of segment $\overline{AB}$. The set consisting of all points of the segment except the endpoints A and B is the *interior* of the segment. The measure of the distance between A and B is the *length* of the segment. Segments with the same length are *congruent* segments. If $\overline{AB}$ and $\overline{CD}$ are congruent, we write $\overline{AB} \cong \overline{CD}$.

Note. The proof of the next result is to be given in Exercise 2.5.5. It follows from Theorem 2.5.1 and Definition 2.5.2.

Theorem 2.5.2. If A and B are two points on a line ℓ and if, in any coordinate system on ℓ, A and B have coordinates a and b such that $a < b$, then the segment $\overline{AB}$ is the same as the set of points whose coordinates x satisfy the relation $a \leq x \leq b$. If $b < a$, then the segment $\overline{AB}$ is the same as the set of points whose coordinates satisfy the relation $b \leq x \leq a$.

Definition 2.5.3. If A and B are two points then the set consisting of all points of the segment $\overline{AB}$ and all points P such that B is between A and P is called a *ray*. The ray determined by points A and B (in that order) is denoted $\overrightarrow{AB}$ and point A is the *endpoint* of ray $\overrightarrow{AB}$. Two rays with the same endpoint are *concurrent*. Two concurrent rays which are collinear are *opposite rays* and each is *opposite* to the other.

Note. The next theorem allows us to give a ray in terms of coordinates.

Theorem 2.5.3. Let A and B be distinct points and let a and b be, respectively, the coordinates of these points in any coordinate system on $\overleftrightarrow{AB}$. Then if $a < b$, the ray $\overrightarrow{AB}$ is the same as the set of points whose coordinates x satisfy the condition $a \leq x$. If $a > b$, the ray $\overrightarrow{AB}$ is the same as the set of points whose coordinates satisfy the condition $a \geq x$.

Note. The next result is related to Theorem 2.5.3 and a proof is to be given in Exercise 2.5.6.

Theorem 2.5.4. Let P be any point on an arbitrary line ℓ and let p be the coordinate of P in any coordinate system on ℓ. Then the set of points of ℓ whose coordinates x satisfy the condition $x \leq p$ and the set of points whose coordinates satisfy the condition $x \geq p$ are opposite rays on ℓ with common endpoint P.

Note. The next result is reminiscent of Euclid's compass and straightedge approach. It gives us two points on a line that are a given distance from a given point.

Theorem 2.5.5. The Point-Plotting Theorem. If $\overrightarrow{AB}$ is a ray and d a positive number, then there is exactly one point on $\overrightarrow{AB}$ and one point on the ray opposite to $\overrightarrow{AB}$ such that the distance from A to each of these points relative to a given unit pair is d.

Note. If we take d in Theorem 2.5.5 to be $k \cdot AB$ where k is positive, then we get the following theorem. Details are to be given in Exercise 2.5.7.

Theorem 2.5.6. If A and B are any two points and if k is an arbitrary positive number, then there is a unique point P on $\overrightarrow{AB}$ such that $AP = k \cdot AB$. Moreover, is the coordinates of A and B in a coordinate system on $\overleftrightarrow{AB}$ are a and b, respectively, then the coordinate of P is the number $p = a + k(b - a)$.

Note. With $k = 1/2$ in Theorem 2.5.6, we get the following special case. We will use it to define the midpoint of a segment.

Corollary 2.5.1. There is a unique point P between two given points, A and B, such that $AP = PB$. Moreover, if the coordinates of A and B are a and b, respectively, then the coordinate of P is $p = (a + b)/2$.

Definition. The point P of Corollary 2.5.1 such that $AP = PB$ is the *midpoint* of the segment $\overline{AB}$. The midpoint of a segment is said to *bisect* the segment and, more generally, any set of points whose intersection with a segment consists only of the midpoint of the segment is said to *bisect* the segment.

Note 2.5.A. From Theorems 2.5.3 and 2.5.4 we have that for any point A on a line, there are two opposite rays that have A as their common endpoint. These two rays (excluding point A) can be interpreted as the two "sides" of A on the line. These sides satisfy:

(1) any segment such as $\overline{BC}$, whose endpoints are on the same side of A lies entirely on that side of A, and

(2) any segment such as $\overline{CD}$ whose endpoints lie on opposite sides of A contains A in its interior.

See Figure 2.7. This idea of "sides" will be carried into (2-dimensional) planes and (3-dimensional) space below when we consider halfplanes and halfspaces, though this will require another postulate.

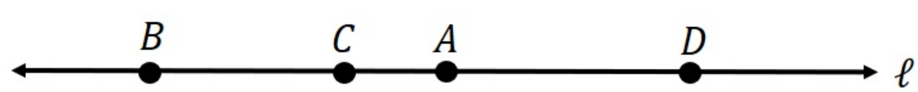

Figure 2.7

To explore the ideas of sides of a plane in space, we first introduce the idea of a "convex set" and revisit the sides of a point on a line in terms of convex sets.

Definition 2.5.4. A set of points is *convex* if any segment whose endpoints belong to the set lies entirely in the set.

Note. Figure 2.8 gives two convex sets (the triangle and circle) and one non-convex set.

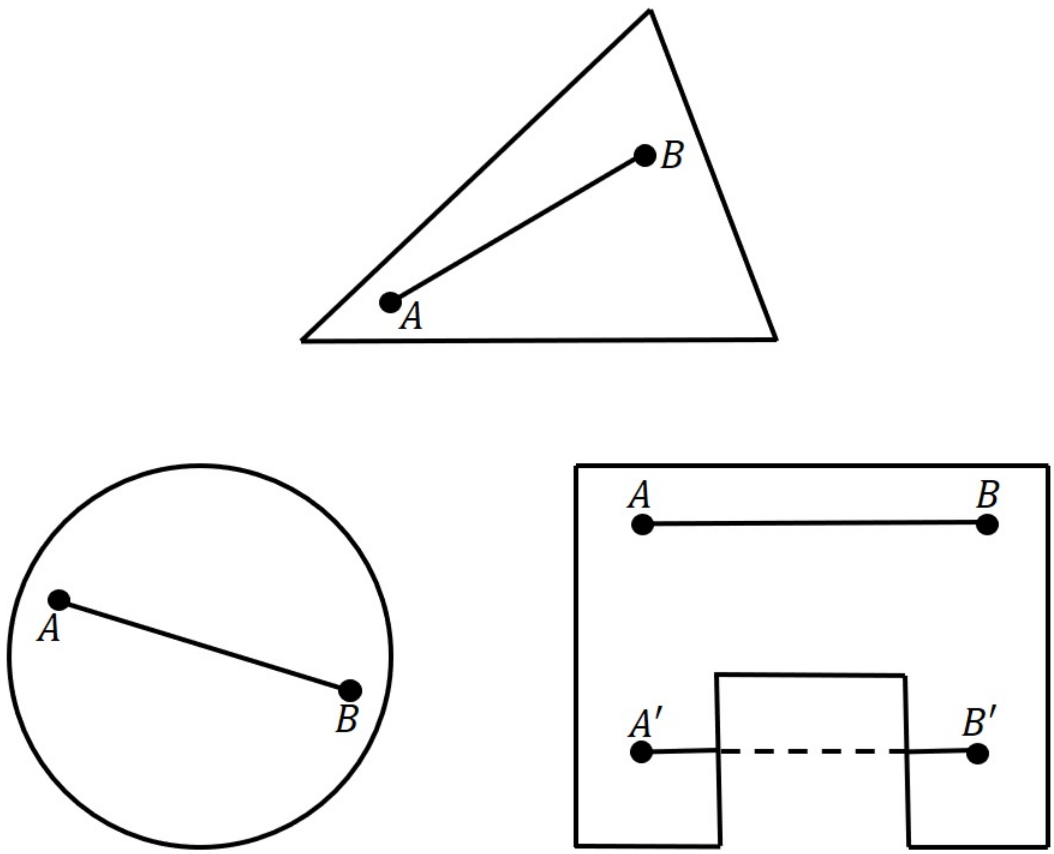

Figure 2.8

Theorem 2.5.7. The intersection of two convex sets is a convex set.

Note. Note 2.5.A can be stated in the terminology of convex sets as follows.

Theorem 2.5.8. Any point A divides the rest of the points on any line containing A into two classes such that:

(1) each set is convex, and

(2) any segment joining a point in one set to a point in the other contains A in its interior.

Note. We now state a postulate that effectively mimics Theorem 2.5.8, but "raises things by one dimension." Instead of separating a line into two convex rays with a point, it addresses separating a plane into two convex sets using a line.

Postulate 12. The Plane-Separation Postulate. For any plane and any line lying in the plane, the points of the plane which do not belong to the line form two sets such that:

(1) each set is convex, and

(2) any segment joining a point in one set to a point in the other intersects the given line.

Definition. Each of the convex sets in a given plane, as described in Postulate 12, is a *halfplane*. The line of Postulate 12 is the *edge* of each halfplane (and the line does not belong to either halfplane, as postulated) and is said to *separate* the plane into the two halfplanes. Points in the same halfplane are said to lie on the *same side* of the given line, and two points which lie in different halfplanes are said to lie on *opposite sides* of the given line.

Note. Surprisingly (I think), we do *not* need an additional postulate to extend the idea of separation up another dimension. We can use the Plane Separation Postulate (Postulate 12) to show that a plane separates (3-dimensional) space into two convex sets, similar to Theorem 2.5.8 (for the separation of a line with a point) and Postulate 12 (for the separation of a plane with a line). In fact, these ideas can be generalized to higher dimensions. The n-dimensional space you consider sophomore Linear Algebra (MATH 2010) can be separated by a "hyperplane." See my onlin notes for Linear Algebra on Section 2.5. Lines, Planes, and Other Flats where a hyperplane in $\mathbb{R}^n$ is defined as an $(n-1)$-flat. Hyperplanes are address in a setting more general than a finite dimensional vector space in ETSU's Fundamentals of Functional Analysis (MATH 5740). See my online notes for this class on Section 5.5. Geometric Versions of Hahn-Banach Theorem where hyperplanes are defined as translations of maximal proper subspaces of a vector space. In this setting, the vector space is separated into two halfspaces, analogous to our situation here.

Theorem 2.5.9. The points of space which do not lie in a given plane form two sets such that:

(1) each set is convex, and

(2) any segment joining a point in one set to a point in the other set intersects the given plane.

Definition. Each of the convex sets determined by the given plane in Theorem 2.5.9 is a *halfspace*. The plane itself is a *face* of each halfspace (the plane does intersect either halfspace by definition) and is said to *separate* the space into two halfspaces. Points in the same halfspace are said to lie on the *same side* of the given plane, and two points which lie in different halfspaces are said to lie on *opposite sides* of the given plane.

Note. The next theorem is an application of the Plane-Separation Postulate (Postulate 12). It will be used in Section 2.7. Further Properties of Angles (see the proof of Theorem 2.7.1).

Theorem 2.5.10. If V is any point on the edge of a halfplane H and if A, B, and X are three points in the union of H and its edge such that:

(1) no two of the points A, B, X are collinear with V and

(2) A and B lie on opposite sides of $\overleftrightarrow{VX}$,

then A and X lie on the same side of $\overleftrightarrow{VB}$, and B and X lie on the same side of $\overleftrightarrow{VA}$.

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