2.6. Angles and Angle Measurement

Note. Wylie describes two different concepts of an angle (see page 68). One view is an angle as an amount of rotation and the other is an angle as a set of points. The definition we use is based on the second view, and the first view is dealt with in terms of the measure of an angle. In this section, we define these ideas and in the process introduce three new postulates.

Definition 2.6.1. An angle is the union of two rays which have a common endpoint and do not lie on the same straight line. Each of the rays is a *side* of the angle, and the common endpoint of the two rays is the *vertex* of the angle.

Note. The angle formed by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted as $\angle BAC$ or $\angle CAB$. If there can be no ambiguity, then we denote this angle as $\angle A$ (as we will often do when dealing with triangles). Notice that Definition 2.6.1 does not consider "zero angles" or "straight angles." We will not put a direction on angles, and we'll see below when we introduce measures of angles that (in degrees) the measure of an angle is always between $0°$ and $180°$. We next give three postulates on the measurement of angles, which are similar to the postulates for the measurement of distance in Section 2.4. The Measurement of Distance.

Postulate 13. If R is any positive number, there exists a correspondence which associates with each angle in space a unique positive number between 0 and R.

Definition. The number R of Postulate 13 is the *scale factor*. The number assigned to a given angle in Postulate 13 is the *measure of the angle relative to the scale factor* R and for angle $\angle BAC$ is denoted $m_R\angle BAC$.

Note. The scale factor R represents the measure of a "straight angle." For example, when $R = 180$ we are measuring angles in degrees. When $R = \pi$ we are measuring degrees in radians. The next postulate implies that we can convert from one scale factor to another (similar to Postulate 9 which allowed us to convert the measure of distance between two unit pairs).

Postulate 14. If R and S are any positive numbers, then for every angle $\angle BAC$, we have $m_R\angle BAC = \dfrac{R}{S} m_s\angle BAC$.

Note. The last postulate of this section allows us to set up a "theoretical protractor" which can be used to arithmetically measure any angle. See Figure 2.13 below.

Postulate 15. The Protractor Postulate. If H is any halfplane, $\overrightarrow{VA}$ any ray lying in the edge of H, and R any positive number, there is a one-to-one correspondence between the set of all numbers x for which $0 \le x \le R$ and the set of rays, $\overrightarrow{VX}$ which lie in the union of H and its edge, such that:

(1) $\overrightarrow{VA}$ corresponds to the number 0,

(2) the ray opposite to $\overrightarrow{VA}$ corresponds to the number R, and

(3) if X and Y are not collinear with V and if x and y are the numbers which correspond to $\overrightarrow{VX}$ and $\overrightarrow{VY}$, respectively, then $m_R\angle XVY = |x - y|$.

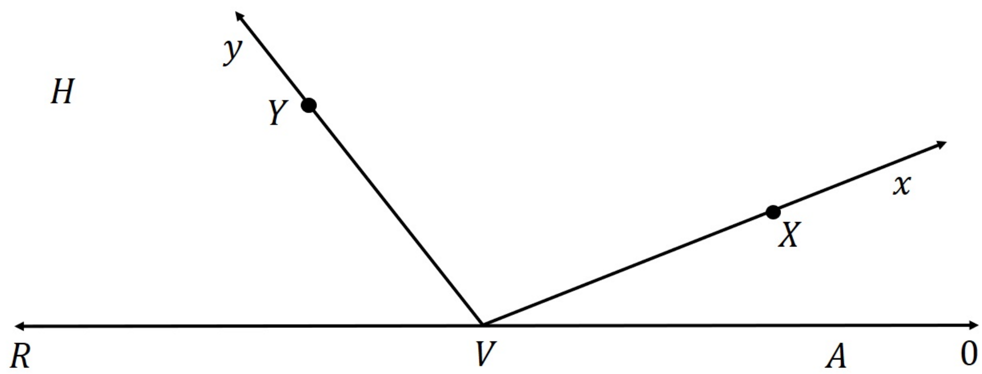

Figure 2.13

Note. Analogous to Theorems 2.4.1 and 2.4.2 on the addition and ratio of measures of distances, we have the following two theorems on measures of angles. Proofs are to be given in Exercises 2.6.1 and 2.6.2.

Theorem 2.6.1. If $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VC}$ are rays such that for some scale factor R, $m_R\angle AVB + m_R\angle BVC = m_R\angle AVC$ then for any other scale face S, $m_S\angle AVB + m_S\angle BVC = m_S\angle AVC$.

Theorem 2.6.2. If $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, and $\overrightarrow{VD}$ are rays such that for some scale factor R, $\dfrac{m_R\angle AVB}{m_R\angle CVD} = k$ then for any other scale factor S, $\dfrac{m_S\angle AVB}{m_S\angle CVD} = k$.

Note. Analogous to Theorem 2.4.5 (The Point-Plotting Theorem) which allowed us to use a given distance to find a point on a ray, the next theorem allows us to construct angles of a given measure. Its proof is to be given in Exercise 2.6.3, but we include it here as an example.

Theorem 2.6.3. The Angle-Construction Theorem. If H is a halfplane whose edge contains the ray $\overrightarrow{VA}$ and if r is any number (strictly) between 0 and R, there is a unique ray $\overrightarrow{VX}$ such that X is in H and $m_R\angle AVX = r$.

Note. In much of what follows, the value of R will be understood to be given and so we will often denote measures simply with an m so that we denote $m_R\angle AVB$ and $m\angle AVB$. In the next definition, we put an idea of betweenness on concurrent coplanar rays. Notice that the definition implies that the sum of the measures of the two smaller angles is less than 180°. This restriction is necessary because, for example, if three concurrent coplanar rays are at and angle of 120° to each other, then we cannot state that one is between the other two. Notice the similarity in the next definition and the definition of "between" for points in Definition 2.5.1.

Definition 2.6.2. If three consecutive coplanar rays, $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$, are such that

(1) no two of the points A, B, C all lie in or on the edge of a halfplane whose edge passes through V,

(2) A, B, and C all lie in or on the edge of a halfplane whose edge passes through V, and

(3) $m\angle AVB + m\angle BVC = m\angle AVC$,

then $\overrightarrow{VB}$ is said to *lie between* $\overrightarrow{VA}$ and $\overrightarrow{VC}$.

Note. The next result relates the "lies between" idea to the number assigned to an angle by the Protractor Postulate (Postulate 15). This result is similar to Theorem 2.5.1 that relates betweenness of collinear points to the betweenness of their coordinates. The proof is to be given in Exercise 2.6.5.

Theorem 2.6.4. If $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VC}$ are three concurrent coplanar rays such that no two of the points A, B, C are collinear with V, then $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ if and only if under some correspondence described by the Protractor Postulate (Postulate 15), the number assigned to $\overrightarrow{VB}$ is between the numbers assigned to $\overrightarrow{VA}$ and $\overrightarrow{VC}$.

Note. The next result gives the existence of an angle of k times the measure of any existing angle, where $0 < k < 1$. The proof is to be given in Exercise 2.6.6.

Theorem 2.6.5. If $\angle AVC$ is an angle and if k is any number between 0 and 1, there is a unique ray $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ such that $m\angle AVB = k \cdot m\angle AVC$. Moreover, if the numbers assigned to $\overrightarrow{VA}$ and $\overrightarrow{VC}$ in some correspondence described by the Protractor Postulate (Postulate 15) are a and c, respectively, the number assigned to $\overrightarrow{VB}$ is $b = a + k(c - a)$.

Corollary 5.6.1. If $\angle AVC$ is any angle, there is a unique ray $\overrightarrow{VB}$ between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ such that $m\angle AVB = m\angle BVC$. Moreover, if the numbers assigned to $\overrightarrow{VA}$ and $\overrightarrow{VC}$ by some correspondence described by the Protractor Postulate (Postulate 15) are a and c, respectively, then the number assigned to $\overrightarrow{VB}$ is $b = (a + c)/2$.

Note. We now see that the Protractor Postulate (Postulate 15) results in not only a more rigorous version of geometry than that in Euclid's *Elements*, it also insures the existence of more objects. For example, within the restrictions of the postulates and axioms of Euclid an arbitrary angle cannot be trisected. This is one of the three compass and straightedge constructions that are impossible; see my online notes for Introduction to Modern Geometry—History (MATH 4157/5157) on Section 1.8. Three Famous Problems of Greek Geometry. One of these problems is the trisection of an angle. The proof that this cannot be done within the constraints of Euclid's approach is based on the fact that a 20° angle cannot be constructed; this is shown Introduction to Modern Algebra (MATH 4127/5127) in Section VI.32. Geometric Constructions and in Modern Algebra 2 (MATH 5420) in Section V.1 Appendix. Ruler and Compass Constructions (see "Corollary V.1.17. Straight Edge and Compass Trisection of a General Angle is Impossible" which is stated as: "An angle of 60° cannot be trisected by ruler and compass constructions, and therefore a general angle cannot be trisected."). However, an equilateral triangle is constructed in Euclid's Book I Proposition 1, so that a 60° is constructible in Euclid's approach. With the application of Theorem 2.6.5 when $k = 1/3$, we then have the existence of a 20° angle (and more generally, any angle can be trisected by Theorem 2.6.5 with

$k = 1/3$). Of course the trisection is not attained with a compass and straightedge here, but instead by using the Protractor Postulate. Corollary 5.6.1 allows us to bisect a given angle; Euclid *can* deal with the bisection of an angle in terms of a compass and straightedge and does so in Book I Proposition 9.

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