2.7. Further Properties of Angles

Note. In this section, we give several definitions related to angles. In particular, we define the interior/exterior, congruent angles, right angle, supplementary angles, complementary angles, and acute/obtuse angles. Many of these ideas will be used again in the next few sections concerning triangles.

Definition 2.7.1. The *interior* of an angle $\angle AVB$ is the set of all points X such that the ray $\overrightarrow{VX}$ lies between the ray $\overrightarrow{VA}$ and the ray $\overrightarrow{VB}$.

Theorem 2.7.1. The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and B and the halfplane determined by $\overleftrightarrow{VB}$ and A.

Theorem 2.7.2. The interior of an angle is a convex set.

Theorem 2.7.3. If on each side of an angle a point other than the vertex is selected, every point between these points is in the interior of the angle.

Definition 2.7.2. The *exterior of an angle* is the set of all points in the plane of the angle which do not belong to the angle and do not lie in the interior of the angle.

Definition 2.7.3. Angles which have the same measure are said to be *congruent.* If angles $\angle ABC$ and $\angle DEF$ are congruent then we denote this as $\angle ABC \cong \angle DEF$.

Note. Since an angle is a set of points, we do not say that different angles are "equal," but instead say that they are "congruent" (provided their measures are equal); it is the measures that are equal, not the angles.

Definition 2.7.4. Three concurrent rays, two of which are opposite rays, are said to form a *linear pair* of angles.

Definition 2.7.5. Two coplanar angles such that:

(1) they have a common vertex,

(2) they have one ray in common, and

(3) the intersection of their interiors is the empty set

are *adjacent angles.*

Note. In these notes, we take $R = 180$ and so will measure angles in degrees (though we will suppress the degrees symbol, $\circ$). This allows us to give several definitions in terms of the measure of angles.

Definition. Two angles whose measures add to 180 are *supplementary angles* and each is a *supplement* of the other. An angle whose measure is 90 is a *right angle.*

Two angles whose measures add to 90 are *complementary angles* and each is a *complement* of the other. An angle whose measure is less than 90 is an *acute angle*. An angle whose measure is more than 90 is an *obtuse angle*. Two angles whose sides form two pairs of opposite rays are *vertical angles*.

Note. As a quick comment, the term "complementary" gives the name to the trigonometric cofunctions. For example, if angles α and β are complementary, then $\sin\alpha = \cos\beta$. It is the "co" in **co**mplementary that puts the "co" in **co**sine (and in **co**tangent and **co**cosecant).

Note. We can use the definitions above to easily prove the following six theorems, the proofs of which are to be given in Exercises 2.7.6 through 2.7.11, respectively.

Theorem 2.7.4. If two angles form a linear pair, they are supplementary.

Theorem 2.7.5. If the two angles of a linear pair congruent, each is a right angle.

Theorem 2.7.6. Supplementary adjacent angles form a linear pair.

Theorem 2.7.7. Supplements of congruent angles are congruent.

Theorem 2.7.8. Complements of congruent angles are congruent.

Theorem 2.7.9. Vertical angles are congruent.

Definition 2.7.6. The lines determined by two rays which form a right angle are *perpendicular lines*.

Definition 2.7.7. Two sets, each of which is a segment, a ray, or a line which determines two perpendicular lines, are *perpendicular sets*. If the two sets are S_1 and S_2, then we denote this as $S_1 \perp S_2$.

Note. The next result considers the existence of a perpendicular to a line at a given point. This is addressed in Euclid's *Elements* as Book I, Proposition 11.

Theorem 2.7.10. At each point of a given line there is one and only one line which is perpendicular to the given line and lies in a given plane containing the line.

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