2.8. Triangles and Polygons

Note. In this section, we define triangles and their components, quadrilaterals and their components, and polygons and their components. We consider interiors of triangles and polygons in terms of convexity.

Definition 2.8.1. The union of the three segments determined by three non-collinear point is a *triangle*. If the points are A, B, C, then the triangle is denoted $\triangle ABC$. Points A, B, C are the *vertices* of $\triangle ABC$. The segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are the *sides* of $\triangle ABC$. A triangle one of whose angles is a right angle is a *right triangle*. A triangle two of hose sides are congruent is an *isosceles triangle*. If the three sides of a triangle are congruent then the triangle is an *equilateral triangle*. Any side of a triangle and the angle whose vertex is not a point of that side are *opposite* each other. In a right triangle, the side opposite the vertex of the right angle is the *hypotenuse* and the other two sides are the *legs*. For any triangle, the perpendicular segment from any vertex to the line determined by the opposite side is an *altitude*.

Note. Since an angle is, by definition, the set of points on the two concurrent rays that make up the angle, a triangle cannot "contain" an angle (in the sense of set inclusion). So we refer to "the angles of a triangle" or "the angles determined by a triangle," but not to the angles "contained" in a triangle.

Definition 2.8.2. The interior of a triangle is the intersection of the interiors of the three angles of the triangle.

Definition 2.8.3. The exterior of a triangle is the set of all points in the plane of the triangle which do not belong either to the triangle or to its interior.

Theorem 2.8.1. The interior of a triangle is a convex set.

Note. Since the interior of an angle is a convex set by Theorem 2.7.2, and the intersection of two convex sets is convex by Theorem 2.5.7, then the proof of Theorem 2.8.1 easily follows.

Definition 2.8.4. If A, B, C, and D are four coplanar points such that:

(1) no three of the points are collinear and

(2) none of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersects any other at a point which is not one of its endpoints,

then the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ is a *quadrilateral*. The points A, B, C, D are the *vertices* of the quadrilateral. The four segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are the *sides* of the quadrilateral. Two vertices of a quadrilateral which are endpoints of the same side are *consecutive vertices*; two vertices which are not consecutive are *opposite vertices*. Two sides of a quadrilateral which have a common endpoint are *consecutive sides*; two sides which are not consecutive are *opposite sides*. The segments joining opposite vertices of a quadrilateral are the *diagonals* of the quadrilateral.

Note. Unlike with triangles, a quadrilateral is not uniquely determined by its vertices (see Figure 2.23).

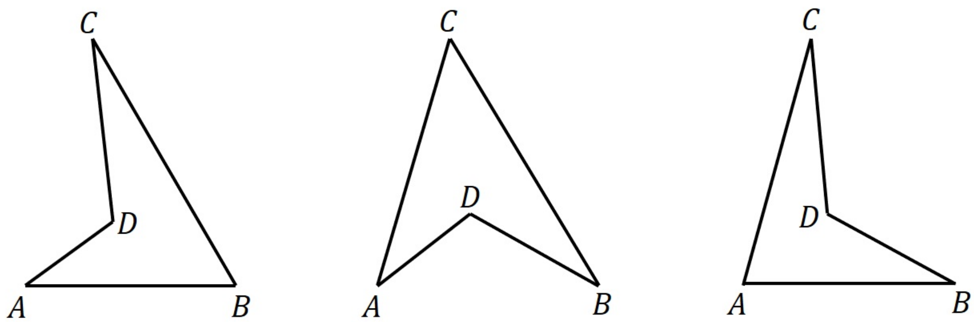

Figure 2.23. Different quadrilaterals with the same vertices.

Notationally, we denote a quadrilateral by presenting the four vertices in an ordered list such that

(1) adjacent vertices in the list are endpoints of the same side, and

(2) the first and last vertices in the list are endpoints of the same side.

So the quadrilateral with vertices A, B, C, D and sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ (see Figure 2.23 left) is denoted: Quadrilateral $ABCD$. Of course we can cycle the order of the vertices around and get four other notations for the same quadrilateral.

Note. The next definition of a "convex quadrilateral" may initially seem a bit odd, since we already have a notion of a convex set. We will show in Theorem 2.8.2 below that this definition is consistent with the definition of convex set (Definition 2.5.4) given previously.

Definition 2.8.5. A quadrilateral is a *convex quadrilateral* if and only if each of its sides lies in the edge of a halfplane which contains the rest of the quadrilateral.

Definition 2.8.6. The *interior* of a convex quadrilateral is the intersection of the halfplanes, each of which has a side of the quadrilateral in its edge and contains the rest of the quadrilateral.

Note. Since halfplanes are convex sets by definition (see Postulate 12), and the intersection of two convex sets is convex by Theorem 2.5.7, then we can easily conclude that the interior of a convex quadrilateral is, in fact, a convex set, as follows.

Theorem 2.8.2. The interior of a convex quadrilateral is a convex set.

Definition. In a convex quadrilateral, the angles determined by pairs of consecutive sides are the *angles of the quadrilateral.* Two angles of a convex quadrilateral are *consecutive angles* is they have a side of the quadrilateral in common. Two angles of a convex set which are not consecutive angles are *opposite angles.* A convex quadrilateral each of whose angles is a right angle is a *rectangle.*

Definition 2.8.7. If $P_1, P_2, \ldots, P_n$ are $n \geq 3$ coplanar points and if the n segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\ldots$, $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ are such that:

(1) no two segments with a common endpoint are collinear,

(2) no two segments intersect except possible at their endpoints,

then the union of the segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\dots$, $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ is a *polygon*. The n points $P_1, P_2, \dots, P_n$ are the *vertices* of the polygon. The n segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\dots$, $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ are the *sides* of the polygon.

Note. We define consecutive and nonconsecutive vertices, and consecutive and nonconsecutive sides of a polygon in the same was as for quadrilaterals. If a polygon has more than four sides, then the concepts of opposite vertices and opposite sides are not defined. As with quadrilaterals, we can denote a polygon with an ordered list of the vertices under the conventions that:

(1) vertices which are adjacent in the list are endpoints of the same side,

(2) the first and last vertices are endpoint of the same side.

Definition. A polygon is a *convex polygon* if and only if each of its sides lies in the edge of a halfplane which contains all the rest of the polygon. The *interior* of a convex polygon is the intersection of all of the halfplanes, each of which has a side of polygon in its edge and contains the rest of the polygon. Then *angles* of a convex polygon are the angles determined by pairs of consecutive sides. A convex polygon whose angles are all congruent and whose sides are all congruent is a *regular polygon*.

Note. We have not defined the interior of a non-convex polygon. Nor have we defined the angles of a non-convex polygon (remember that, by definition, all angles are of measure 0° to 180° and this impacts our ability to define interior angles of non-convex polygon).

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