2.9. The Congruence Postulate

Note. In this section, we define congruence of triangles, postulate that "side-angle-side" results in congruent triangles, and prove that angle-side-angle (Theorem 2.9.1) and side-side-side (Theorem 2.9.4) result in congruent triangles. We also prove that for a given point not on a given line, there is one and only one line perpendicular to the given line (Theorem 2.9.5). This gives the existence of an altitude of a triangle, as defined in Definition 2.8.1.

Definition 2.9.1. A one-to-one correspondence between the vertices of the same or different triangles such that corresponding sides and corresponding angles are congruent is a *congruence* between the triangles.

Definition 2.9.2. Triangles are *congruent* if and only if there exists a congruence between them.

Note. If $\triangle ABC$ and $\triangle DEF$ are congruent, say with the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$, then we write $\triangle ABC \cong \triangle DEF$. Notice that this may hold, whereas we could have $\triangle ABC \ncong DFE$. We need a postulate which gives the congruence of triangles and relates both angles and sides. This will allow us to prove other conditions under which triangles are congruent. The Congruence Postulate below states that if each of two corresponding sides and the included angle are congruent between two triangles, then the two triangles are congruent. This is usually called "side-angle-side" in a high school geometry class.

Postulate 16. The Congruence Postulate. If there exists a one-to-one correspondence between two triangles or between a triangle and itself in which two sides and the angle determined by these sides in one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence and the triangles are congruent.

Note. We now prove some congruence theorems. The first result is called "angle-side-angle" in high school.

Theorem 2.9.1. If there exists a one-to-one correspondence between two triangles or between a triangle and itself in which two angles and the side common to the two angles in one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence and the triangles are congruent.

Note. The next result deals with isosceles triangles in which two sides are congruent. This allows for the congruence of a triangle with itself, where the correspondence is not the identity correspondence (that is, such a triangle has a non-identity "symmetry").

Theorem 2.9.2. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then $\triangle ABC \cong \triangle ACB$ and $\angle ABC \cong \angle ACB$.

Note. The next theorem and its corollary concern perpendiculars to a side of an isosceles triangle.

Theorem 2.9.3. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then the segment determined by A and the midpoint of $\overline{BC}$ is perpendicular to $\overline{BC}$.

Corollary 2.9.1. The perpendicular bisector of the base of an isosceles triangle passes through the vertex opposite that side.

Note. The next result concerning congruent triangles is called "side-side-side" in high school.

Theorem 2.9.4. If there exists a one-to-one correspondence between two triangles, or between a triangle and itself, in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent.

Note. The last result again concerns the existence of perpendicular lines. As commented above, it gives the existence of an altitude of a triangle, as defined in Definition 2.8.1.

Theorem 2.9.5. Through a given point not on a given line there is one and only one line perpendicular to the given line.