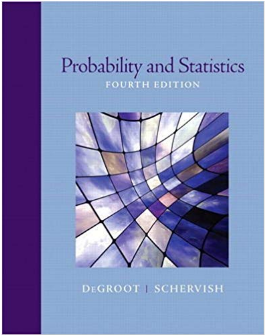
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.10. The Probability of a Union of Events—Proofs of Theorems



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Theorem 1.10.1

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$$\Pr(A_1 \cup A_2 \cup A_3) = \Pr(A_1) + \Pr(A_2) + \Pr(A_3) - \Pr(A_1 \cap A_2)$$
$$-\Pr(A_2 \cap A_3) - \Pr(A_1 \cap A_3) + \Pr(A_1 \cap A_2 \cap A_3).$$

Proof. For two events $A = A_1 \cup A_2$ and $B = A_3$ we have

$$\begin{aligned}
& \Pr(A_1 \cup A_2 \cup A_3) \\
= \; & \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \text{ by Theorem 1.5.7} \\
= \; & \Pr(A_1 \cup A_2) + \Pr(A_3) - \Pr((A_1 \cup A_2) \cap A_3) \\
= \; & \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2) + \Pr(A_3) - \Pr((A_1 \cup A_2) \cap A_3) \\
& \text{ by Theorem 1.5.7} \\
= \; & \Pr(A_1) + \Pr(A_2) + \Pr(A_3) - \Pr(A_1 \cap A_2) - \Pr((A_1 \cap A_3) \cup (A_2 \cap A_3)) \\
& \text{ by Theorem 1.4.10}
\end{aligned}$$

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Theorem 1.10.1 (continued)

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Proof (continued).

$$\Pr(A_1 \cup A_2 \cup A_3)$$

$$\begin{aligned} &= \Pr(A_1) + \Pr(A_2) + \Pr(A_3) - \Pr(A_1 \cap A_2) \\ &\quad - (\Pr(A_1 \cap A_3) + \Pr(A_2 \cap A_3) - \Pr(A_1 \cap A_2 \cap A_3)) \text{ by Theorem 1.5.7} \\ &= \Pr(A_1) + \Pr(A_2) + \Pr(A_3) - \Pr(A_1 \cap A_2) - \Pr(A_2 \cap A_3) \\ &\quad -\Pr(A_1 \cap A_3) + \Pr(A_1 \cap A_2 \cap A_3), \end{aligned}$$

as claimed. □

Theorem 1.10.2

Theorem 1.10.2. For any n events $A_1, A_2, \ldots, A_n$ we have

$$\Pr\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i) - \sum_{i<j} \Pr(A_i \cap A_j) + \sum_{i<j<k} \Pr(A_i \cap A_j \cap A_k)$$

$$- \sum_{i<j<k<\ell} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) + \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n).$$

Proof. The result holds trivially for $n = 1$, holds by Theorem 1.5.7 for $n = 2$, and holds by Theorem 1.10.1 for $n = 3$. We now prove the result by mathematical induction. Suppose the result holds for all $n \le m$. Let $A_1, A_2, \ldots, A_{m+1}$ be events. Define $A = \cup_{i=1}^{m} A_i$ and $B = A_{m+1}$. Then

$$\begin{aligned} \Pr\left(\cup_{i=1}^{m+1} A_i\right) &= \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &\quad \text{by Theorem 1.5.7} \\ &= \Pr\left(\cup_{i=1}^{m} A_i\right) + \Pr(A_{m+1}) - \Pr\left(\left(\cup_{i=1}^{m} A_i\right) \cap A_{m+1}\right) \end{aligned}$$

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$$\begin{aligned}
\Pr\left(\cup_{i=1}^{m+1} A_i\right) &= \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \\
&\quad \text{ by Theorem 1.5.7} \\
&= \Pr\left(\cup_{i=1}^{m} A_i\right) + \Pr(A_{m+1}) - \Pr\left(\left(\cup_{i=1}^{m} A_i\right) \cap A_{m+1}\right)
\end{aligned}$$

Theorem 1.10.2 (continued 1)

Proof (continued). ...

$$= \Pr\left(\cup_{i=1}^{m} A_i\right) + \Pr(A_{m+1}) - \Pr\left(\cup_{i=1}^{m}(A_i \cap A_{m+1})\right)$$

by Theorem 1.4.10, Distributive Property, and induction

$$= \left(\sum_{i=1}^{m} \Pr(A_i) - \sum_{i<j\le m} \Pr(A_i \cap A_j) + \sum_{i<j<k\le m} \Pr(A_i \cap A_j \cap A_k)\right.$$

$$- \sum_{i<j<k<\ell\le m} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) + \cdots$$

$$\left. + (-1)^{m+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_m)\right) + \Pr(A_{m+1})$$

$$- \left(\sum_{i=1}^{m} \Pr(A_i \cap A_{m+1}) - \sum_{i<j\le m} \Pr(A_i \cap A_j \cap A_{m+1}) \ldots\right.$$

Theorem 1.10.2 (continued 2)

Proof (continued). ...

$$+\sum_{i<j<k\le m} \Pr(A_i \cap A_j \cap A_k \cap A_{m+1})$$

$$-\sum_{i<j<k<\ell\le m} \Pr(A_i \cap A_j \cap A_k \cap A_\ell \cap A_{m+1}) + \cdots$$

$$\left. + (-1)^{m+1}\Pr(A_1 \cap A_2 \cap \cdots \cap A_m \cap A_{m+1})\right) \text{ by induction hypothesis}$$

$$= \sum_{i=1}^{m+1} \Pr(A_i) - \sum_{i<j\le m+1} \Pr(A_i \cap A_j) + \sum_{i<j<k\le m+1} \Pr(A_i \cap A_j \cap A_k)$$

$$-\sum_{i<j<k<\ell\le m+1} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) + \cdots$$

$$+(-1)^{m+2}\Pr(A_1 \cap A_2 \cap \cdots \cap A_m \cap A_{m+1}),$$

so result holds for $n = m+1$ and, by induction, holds for $n \in \mathbb{N}$. □