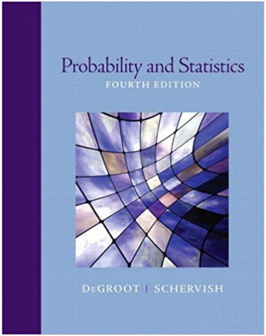
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.5. The Definition of Probability—Proofs of Theorems



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Theorem 1.5.1

Theorem 1.5.1. $\Pr(\varnothing) = 0$.

Proof. Let $A_i = \varnothing$ for $i \in \mathbb{N}$. Since $A_i \cap A_j = \varnothing$ for any $i, j \in \mathbb{N}$ then this sequence of events is an infinite disjoint sequence and so by Axiom 3, Axiom of Countability, we have (since $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} \varnothing = \varnothing$):

$$\Pr(\varnothing) = \Pr\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{pr}(A_i) = \sum_{i=1}^{\infty} \Pr(\varnothing).$$

But the only real number a such that $a = \sum_{i=1}^{\infty} a$ is $a = 0$. Therefore $\Pr(\varnothing) = 0$, as claimed. □

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Theorem 1.5.2

Theorem 1.5.2. Finite Additivity. For any finite sequence of n disjoint events $A_1, A_2, \ldots, A_n$ we have $\Pr\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$.

Proof. Consider the infinite sequence of events $A_1, A_2, \ldots$ in which $A_1, A_2, \ldots, A_n$ are the n given disjoint events and $A_i = \varnothing$ for $i > n$. Then $A_1, A_2, \ldots$ is an infinite sequence of disjoint events and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{n} A_i$.

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$$\begin{aligned}
\Pr\left(\cup_{i=1}^n A_i\right) &= \Pr\left(\cup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \Pr(A_i) = \sum_{i=1}^n \Pr(A_i) + \sum_{i=n+1}^\infty \Pr(A_i) \\
&= \sum_{i=1}^n \Pr(A_i) + \sum_{i=n+1}^\infty 0 \text{ by Theorem 1.5.1} \\
&= \sum_{i=1}^n \Pr(A_i),
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Theorem 1.5.3

Theorem 1.5.3. Probability of the Complement. For any event A, $\Pr(A^c) = 1 - \Pr(A)$.

Proof. Since A and A^c are disjoint events and $A \cup A^c = S$ (as observed above) then by Theorem 1.5.2, Finite Additivity, $\Pr(S) = \Pr(A) + \Pr(A^c)$. Since $\Pr(A) = 1$ by Axiom 2, Axiom of Total Probability, then $\Pr(A^c) = \Pr(S) - \Pr(A) = 1 - \Pr(A)$, and claimed. □

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Theorem 1.5.4

Theorem 1.5.4. Monotonicity. If $A \subset B$ then $\Pr(A) \leq \Pr(B)$.

Proof. First we prove $B = A \cup (B \cap A^c)$. Of course A and $B \cap A^c$ are disjoint. Suppose $b \in B$. If $b \in A$ then $b \in A \cup (B \cap A^c)$, and if $b \notin A$ then $b \in A^c$ and so $b \in B \cap A^c$ and hence $b \in A \cup (B \cap A^c)$. So $B \subset A \cup (B \cap A^c)$.

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$$\Pr(B) = \Pr(A \cup (B \cap A^c)) = \Pr(A) + \Pr(B \cap A^c).$$

Since $\Pr(B \cap A^c) \geq 0$ by Axiom 1, Axiom of Non-Negativity, then $\Pr(A) \leq \Pr(B)$, as claimed. $\square$

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Theorem 1.5.5. For any event A, $0 \leq \Pr(A) \leq 1$.

Proof. By Axiom 1, Axiom of Non-Negativity, $\Pr(A) \geq 0$. By Axiom 2 $\Pr(S) = 1$, and so by Theorem 1.5.4 $\Pr(A) \leq \Pr(S) = 1$.

Therefore $0 \leq \Pr(A) \leq 1$, as claimed. □

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Therefore $0 \leq \Pr(A) \leq 1$, as claimed. $\square$

Theorem 1.5.6

Theorem 1.5.6. For any two events A and B,

$$\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B).$$

Proof. By Theorem 2.11, events $A \cap B^c$ and $A \cap B$ are disjoint. By Theorem 1.5.2, Finite Additivity,

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c).$$

The claim now follows. $\square$

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Theorem 1.5.7

Theorem 1.5.7. For any two events A and B,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Proof. By Theorem 1.4.11, $A \cup B = B \,\dot{\cup}\, (A \cap B^c)$. So

$$\begin{aligned}
\Pr(A \cup B) &= \Pr(B) + \Pr(A \cap B) \text{ by Theorem 1.5.2, Finite Additivity} \\
&= \Pr(B) + \Pr(A) - \Pr(A \cap B) \text{ by Theorem 1.5.6.}
\end{aligned}$$

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□