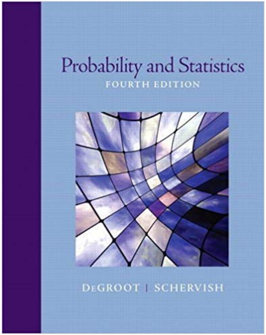
Mathematical Statistics 1

Chapter 2. Conditional Probability

2.1. The Definition of Conditional Probability—Proofs of Theorems



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Theorem 2.1.2

Theorem 2.1.2. Multiplication Rule for Conditional Probabilities. Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \neq 0$. Then

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\Pr(A_4|A_1 \cap A_2 \cap A_3)$$

$$\cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Proof. Since $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \neq 0$ by hypothesis, then by Theorem 1.5.4, "monotonicity," $\Pr(A_1 \cap A_2 \cap \cdots \cap A_i) \geq \Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$ for $i - 1, 2, \ldots, n-1$. So by Theorem 2.1.1, with $A = A_i$ and $B = A_1 \cap A_2 \cap \cdots \cap A_{i-1}$ for $i = 1, 2, \ldots, n$, we have

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_i)}{\Pr(A_1 \cap A_2 \cap \cdots \cap A_{i-1})}.$$

Theorem 2.1.2

Theorem 2.1.2. Multiplication Rule for Conditional Probabilities. Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \neq 0$. Then

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Theorem 2.1.2 (continued)

Theorem 2.1.2. Multiplication Rule for Conditional Probabilities. Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \neq 0$. Then

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\Pr(A_4|A_1 \cap A_2 \cap A_3) \cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

Proof (continued). So

$$\Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\Pr(A_4|A_1 \cap A_2 \cap A_3) \cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

$$= \Pr(A_1)\frac{\Pr(A_1 \cap A_2)}{\Pr(A_1)} \frac{\mathrm{PR}(A_1 \cap A_2 \cap A_3)}{\Pr(A_1 \cap A_2)} \frac{\Pr(A_1 \cap A_2 \cap A_3 \cap A_4)}{\Pr(A_1 \cap A_2 \cap A_3)} \cdots \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n)}{\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1})} = \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

(cancelling numerators with the following denominators), as claimed. □

Theorem 2.1.3

Theorem 2.1.3. Suppose $A_1, A_2, \ldots, A_n, B$ are events such that $\Pr(B) \neq 0$ and $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}|B) \neq 0$. Then

$$\frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n|B)}{\Pr(B)} = \Pr(A_1|B)\Pr(A_2|A_1 \cap B)\Pr(A_3|A_1 \cap A_2 \cap B)$$

$$\cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B).$$

Proof. Since $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}|B) > 0$ then by Theorem 1.5.4, "monotonicity,"
$\Pr(A_1 \cap A_2 \cap \cdots \cap A_i|B) \geq \Pr(A_1, \cap A_2 \cap \cdots \cap A_{n-1}|B) > 0$ for $i = 1, 2, \ldots, n-1$ and we have by Theorem 2.2.1 with $A = A_i$ and $C = A_1 \cap A_2 \cap \cdots \cap A_i \cap B$,

$$\Pr(A|C) = \Pr(A_i|A_1 \cap A_2 \cap \cdots \cap A_{i-1} \cap B) = \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_i \cap B)}{\Pr(A_1 \cap A_2 \cap \cdots \cap A_{i-1} \cap B)}$$

for $i = 1, 2, \ldots, n$.

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$$\frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n|B)}{\Pr(B)} = \Pr(A_1|B)\Pr(A_2|A_1 \cap B)\Pr(A_3|A_1 \cap A_2 \cap B)$$

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Proof. Since $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}|B) > 0$ then by Theorem 1.5.4, "monotonicity,"
$\Pr(A_1 \cap A_2 \cap \cdots \cap A_i|B) \geq \Pr(A_1, \cap A_2 \cap \cdots \cap A_{n-1}|B) > 0$ for $i = 1, 2, \ldots, n-1$ and we have by Theorem 2.2.1 with $A = A_i$ and $C = A_1 \cap A_2 \cap \cdots \cap A_i \cap B$,

$$\Pr(A|C) = \Pr(A_i|A_1 \cap A_2 \cap \cdots \cap A_{i-1} \cap B) = \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_i \cap B)}{\Pr(A_1 \cap A_2 \cap \cdots \cap A_{i-1} \cap B)}$$

for $i = 1, 2, \ldots, n$.

Theorem 2.1.3 (continued)

Theorem 2.1.3. Suppose $A_1, A_2, \ldots, A_n, B$ are events such that $\Pr(B) \neq 0$ and $\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1}|B) \neq 0$. Then

$$\frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n|B)}{\Pr(B)} = \Pr(A_1|B)\Pr(A_2|A_1 \cap B)\Pr(A_3|A_1 \cap A_2 \cap B)$$
$$\cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B).$$

Proof (continued). We have

$$\Pr(A_1|B)\Pr(A_2|A_1 \cap B)\Pr(A_3|A_1 \cap A_2 \cap B) \cdots \Pr(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B)$$
$$= \frac{\Pr(A_1 \cap B)}{\Pr(B)} \frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(A_1 \cap B)} \frac{\Pr(A_1 \cap A_2 \cap A_3 \cap B)}{\Pr(A_1 \cap A_2 \cap B)}$$
$$\frac{\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap B)}{\Pr(A_1 \cap A_2 \cap A_3 \cap B)} \cdots \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n \cap B)}{\Pr(A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B)}$$
$$= \frac{\Pr(A_1 \cap A_2 \cap \cdots \cap A_n \cap B)}{\Pr(B)},$$

as claimed. □

Theorem 2.1.4

Theorem 2.1.4. Law of Total Probability. Suppose the events $B_1, B_2, \ldots, B_k$ form a partition of sample space S and $\text{Pr}(B_j) \neq 0$ for $j = 1, 2, \ldots, k$. Then, for every event A in S, $\text{Pr}(A) = \sum_{j=1}^{k} \text{Pr}(B_j)\text{Pr}(A|B_j)$.

Proof. The events $B_1 \cap A$, $B_2 \cap A$, $B_3 \cap A$, $\ldots$, $B_k \cap A$ form a partition of set A, so by Theorem 1.5.2, "Finite Additivity," $\text{Pr}(A) = \sum_{j=1}^{k} \text{Pr}(B_j \cap A)$. Since $\text{Pr}(B_j) \neq 0$ for $j = 1, 2, \ldots, k$ then by Theorem 2.1.1, $\text{Pr}(B_j \cap A) = \text{Pr}(B_j)\text{Pr}(A|B_j)$ for $j = 1, 2, \ldots, k$. Hence $\text{Pr}(A) = \sum_{j=1}^{k} \text{Pr}(B_j)\text{Pr}(A|B_j)$, as claimed. □

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Proof. The events $B_1 \cap A$, $B_2 \cap A$, $B_3 \cap A$, $\ldots$, $B_k \cap A$ form a partition of set A, so by Theorem 1.5.2, "Finite Additivity," $\Pr(A) = \sum_{j=1}^{k} \Pr(B_j \cap A)$. Since $\Pr(B_j) \neq 0$ for $j = 1, 2, \ldots, k$ then by Theorem 2.1.1, $\Pr(B_j \cap A) = \Pr(B_j)\Pr(A|B_j)$ for $j = 1, 2, \ldots, k$. Hence $\Pr(A) = \sum_{j=1}^{k} \Pr(B_j)\Pr(A|B_j)$, as claimed. □