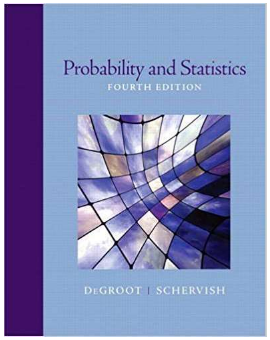
Mathematical Statistics 1

Chapter 2. Conditional Probability

2.2. Independent Events—Proofs of Theorems



Exercise 2.2.2

Exercise 2.2.2. Suppose events A and B are independent. Prove that events A^c and B^c are also independent.

Proof. We know by Theorem 1.5.3, Probability of the Complement, that $\Pr(A^c) = 1 - \Pr(A)$ and $\Pr(B^c) = 1 - \Pr(B)$. So

$$\begin{aligned}
\Pr(A^c)\Pr(B^c) &= (1 - \Pr(A))(1 - \Pr(B)) \\
&= 1 = \Pr(A) = \Pr(B) + \Pr(A)\Pr(B) \\
&= 1 - \Pr(A) - \Pr(B) + \Pr(A \cap B) \text{ since } A \text{ and } B \\
&\quad \text{are independent} \\
&= 1 - (\Pr(A) + \Pr(B) - \Pr(A \cap B)) \\
&= 1 - \Pr(A \cup B) \text{ by Theorem 1.5.6} \\
&= \Pr((A \cup B)^c)\Pr((A \cup B)^c) \text{ by Theorem 1.5.3,} \\
&\quad \text{Probability of the Complement} \\
&= \Pr(A^c \cap B^c) \text{ by Exercise 1.4.4 , DeMorgan's Laws.}
\end{aligned}$$

So, by definition, A^c and B^c are independent, as claimed. □

Theorem 2.2.1

Theorem 2.2.1. If two events A and B are independent, then the events A and B^c are also independent.

Proof. Since $A = (A \cap B^c) \dot{\cup} (A \cap B)$ then by Theorem 1.5.2, Finite Additivity, $\Pr(A) = \Pr(A \cap B^c) + \Pr(A \cap B)$ or $\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B)$. Since A and B are independent then $\Pr(A \cap B) = \Pr(A)\Pr(B)$ and so

$$\begin{aligned}
\Pr(A \cap B^c) &= \Pr(A) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B) \\
&= \Pr(A)(1 - \Pr(B)) \\
&= \Pr(A)\Pr(B^c) \text{ by Theorem 1.5.3,} \\
&\quad \text{Probability of the Complement,}
\end{aligned}$$

and so, by definition, A and B^c are independent. □