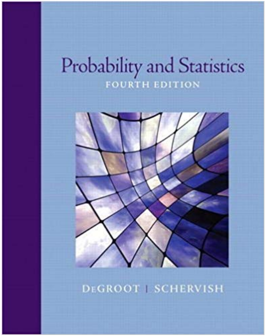
Mathematical Statistics 1

Chapter 2. Conditional Probability

2.2. Independent Events—Proofs of Theorems



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Exercise 2.2.2

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Proof. We know by Theorem 1.5.3, Probability of the Complement, that $\Pr(A^c) = 1 - \Pr(A)$ and $\Pr(B^c) = 1 - \Pr(B)$. So

$$\begin{aligned}
\Pr(A^c)\Pr(B^c) &= (1 - \Pr(A))(1 - \Pr(B)) \\
&= 1 = \Pr(A) = \Pr(B) + \Pr(A)\Pr(B) \\
&= 1 - \Pr(A) - \Pr(B) + \Pr(A \cap B) \text{ since } A \text{ and } B \\
&\quad \text{are independent} \\
&= 1 - (\Pr(A) + \Pr(B) - \Pr(A \cap B)) \\
&= 1 - \Pr(A \cup B) \text{ by Theorem 1.5.6} \\
&= \Pr((A \cup B)^c)\Pr((A \cup B)^c) \text{ by Theorem 1.5.3,} \\
&\quad \text{Probability of the Complement} \\
&= \Pr(A^c \cap B^c) \text{ by Exercise 1.4.4 , DeMorgan's Laws.}
\end{aligned}$$

So, by definition, A^c and B^c are independent, as claimed. □

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Proof. Since $A = (A \cap B^c) \,\dot{\cup}\, (A \cap B)$ then by Theorem 1.5.2, Finite Additivity, $\Pr(A) = \Pr(A \cap B^c) + \Pr(A \cap B)$ or $\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B)$.

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$$\begin{aligned} \Pr(A \cap B^c) &= \Pr(A) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B) \\ &= \Pr(A)(1 - \Pr(B)) \\ &= \Pr(A)\Pr(B^c) \text{ by Theorem 1.5.3,} \\ &\quad \text{ Probability of the Complement,} \end{aligned}$$

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