Section 1.7. Counting Methods

Note. In this section we present some methods of determining the total number of outcomes of certain experiments without actually listing all the outcomes.

Example 1.7.2. We consider an experiment satisfying:

(1) The experiment is performed in two parts.

(2) The first part of the experiment has m possible outcomes $x_1, x_2, \ldots, x_m$ and (regardless of the first outcome) the second part of the experiment has n possible outcomes $y_1, y_2, \ldots, y_n$.

Then the sample space S of this experiment consists of mn ordered pairs (x_1, y_2), (x_1, y_2), $\ldots (x_m, y_n)$. This illustrates the *multiplication rule.*

Theorem 1.7.2. If an experiment consists of k parts where the number of outcomes of part i is n_i then, by repeated application of the multiplication rule, the sample space of the experiment has $n_1 n_2 \cdots n_k$ outcomes.

Definition 1.7.1/Theorem 1.7.3. For $n \in \mathbb{N}$ we define n *factorial* as $n! = n(N-1)(n-2)\cdots(3)(2)(1)$. We define $0! = 1$. For $n \in \mathbb{N}$ and k an integer such that $0 \le k \le n$ the *number of permutations* of n elements taken k at a time is

$$P_{n,k} = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

(Sometimes this is denoted ${}_nP_k$.)

Note. If we arrange a collection of k items from a set of n distinct items, then there are n choices for the first position, $n-1$ choices for the second position, $\dots$, and $n-k+1$ choices for the kth position. So by the multiplication rule there are $n(n-1)(n-1)\cdots(n-k+1) = P_{n,k}$ different arrangements. So the permutations of n elements taken k at a time describes the number of ways to arrange k objects from a collection of n objects. The text book gives this as Theorem 1.7.4.

Exercise 1.7.2. In how many different ways can the five letters a, b, c, d, and e be arranged?

Solution. We want the number of ways to arrange $k = 5$ things from a set of size $n = 5$. The number of such arrangements is

$$P_{5,5} = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120.$$

Note. An experiment in which a first object is chosen from a set of n objects and the first object is removed from the set, then a second object is chosen from the remaining $n-1$ objects and the second object is removed from the set, and so forth is called *sampling without replacement*. When k such objects are chosen, the experiment has a sample space of size $P_{n,k} = n!/(n-k)!$. If the objects are replaced after each is chosen in the above experiment then this is called *sampling with replacement*. The sample space for sampling with replacement is of size n^k.

Example 1.7.11. Obtaining Different Numbers. A box contains n balls numbered $1, 2, \ldots, n$. A first ball is chosen, its number noted, and the ball is returned to the box. This process is iterated until k numbers have been noted. We now calculate the probability that each of the k balls that are selected have a different number. If $k > n$ then the probability is 0 so we now consider the case $1 \le k \le n$. We represent an outcome as a vector $(x_1, x_2, \ldots, x_k)$. The number of such vectors with all different components is $P_{n,k} = n!/(n-k)!$. The total number of such vectors is n^k and so the sample space is of size n^k. We reasonably take this to be a simple sample space, so the probability that the k selected balls all have different numbers is $P_{n,k}\dfrac{1}{n^k} = \dfrac{n!}{(n-k)!n^k}$. Notice that the complement of the event "all have different numbers" is that at least two numbers are the same. So the probability that at least two numbers are the same is $1 - \dfrac{n!}{(n-k)!n^k}$ by Theorem 1.5.3, Probability of the Complement.

Example. The Birthday Problem. We now compute the probability that, in a group of k people (where $2 \le k \le 365$), at least 2 have the same birthday. We assume that the 365 birthdays (ignoring February 29) yield a simple sample space. This is just a rewording of Example 1.7.3 with $n = 365$. So the probability that k people have all different birthdays is $P_{365,k}/365^k$ and the probability that at least two people have the same birthday is

$$p = 1 - \frac{P_{365,k}}{365^k} = 1 - \frac{365!}{(365-k)!365^k} = 1 - \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{365-k+1}{365}.$$

Surprisingly, at only $k = 23$ the probability of at least one shared birthday is over $1/2$. Table 1.1 gives different values of p in terms of k.

Table 1.1 (extended). The probability p that at least two people in a group of k people have the same birthday.

k	p	k	p
2	0.003	21	0.444
3	0.008	22	0.476
4	0.016	23	0.507
5	0.027	24	0.538
6	0.040	25	0.569
7	0.056	26	0.598
8	0.074	27	0.627
9	0.095	28	0.654
10	0.117	29	0.681
11	0.141	30	0.706
12	0.167	31	0.730
13	0.194	32	0.753
14	0.223	33	0.775
15	0.253	34	0.795
16	0.283	35	0.814
17	0.315	40	0.891
18	0.347	50	0.970
19	0.379	60	0.994
20	0.411	100	0.9999997

Note. Numerical computation of $n!$ yields huge results. DeGroot and Schervish state (without proof) Stirling's formula which gives a way to approximate $n!$ using exponentiation.

Theorem 1.7.5. Stirling's Formula. Let

$$s_n = \frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log(n) - n.$$

Then $\lim_{n\to\infty} |s_n - \log(n!)| = 0$. Put another way,

$$\lim_{n\to\infty} \frac{(1\pi)^{1/2} n^{n+1/2} e^{-n}}{n!} = 1.$$

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