Section 1.8. Combinatorial Methods

Note. We consider two methods of counting: combinations and binomial coefficients.

Definition 1.8.1. Consider a set with n elements. Each subset of size k chosen from this set is called a *combination* of n elements taken k at a time. We denote the number of distinct such combinations by the symbol $C_{n,k}$.

Note. Consider a set of n distinct elements. We want to count the number of different subsets containing k elements of the set where $1 \le k \le n$. Now there are $P_{n,k}$ arrangements of k elements from the set. Since we want to count the different subsets (in which case order does not matter), then each given collection of k objects is counted several times in the $P_{n,k}$ arrangements; in fact, each collection of k elements is counted $P_{k,k} = k!$ times (the number of arrangements of k objects). So the number of combinations of k elements from a set of size n is

$$\frac{P_{n,k}}{P_{k,k}} = \frac{n!}{(n-k)!k!} = C_{n,k}.$$

Therefore we have the following.

Theorem 1.8.1. The number of distinct subsets of size k that can be chosen from a set of size n is

$$C_{n,k} = \frac{P_{n,k}}{P_{k,k}} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)!k!}.$$

Exercise 1.8.4. A box contains 24 light bulbs, of which 4 are defective. If a person selects 4 bulbs from the box at random, without replacement, what is the probability that all 4 bulbs will be defective?

Solution. With $n = 24$ and $k = 4$, we see that there are

$$C_{24,4} = \frac{24!}{20!4!} = \frac{(24)(23)(22)(21)}{24} = (23)(22)(21) = 10,626$$

different ways to choose 4 bulbs from the box of 24. But there is only one of these combinations that contains all 4 defective bulbs. So the probability of selecting all 4 bulbs is 1/10,626. □

Theorem 1.8.2. The Binomial Theorem.

For any real numbers x and y and $n \in \mathbb{N}$ we have

$$(x + y)n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Note. We can prove the Binomial Theorem using mathematical induction.

Note. Suppose a set of size n consists two distinct types of objects, say k red balls and $n - k$ green balls. If we wish to arrange all n of these balls, this is equivalent to choosing the k locations for the red balls (or equivalently, the $n - k$ position for the green balls). So the binomial coefficient $\binom{n}{k} = \binom{n}{n-k}$ describes the number of ways this can be done.

Example 1.8.7. Tossing a Coin. Suppose that a fair coin is to be tossed ten times. We calculate the probability of obtaining exactly three heads. In ten tossed there are 2^{10} possible outcomes. For an outcome to contain exactly three heads, we need to choose the three "positions" corresponding to the tosses yielding heads. There are

$$\binom{10}{3} = \frac{10!}{7!3!} = \frac{(10)(9)(8)}{6} = 120$$

such choices and so of the $2^{10} = 1024$ outcomes, 120 of them contain exactly three heads. So the desired probability is $\binom{10}{3}/2^{10} = 120/1024 = 15/128 \approx 0.1172$. □

Example 1.8.8. Sampling Without Replacement. Suppose that a class contains 15 boys and 30 girls, and that 10 students are to be selected at random for a special assignment. We will calculate the probability p that exactly 3 boys will be selected. The total number of committees that can be formed is $C_{45,10} = \binom{45}{10}$. There are $C_{15,3}\binom{15}{3}$ ways to choose 3 boys and $C_{30,7} = \binom{30}{7}$ ways to create the committee with 7 girls. So by the multiplication rule, there are $\binom{15}{3}\binom{30}{7}$ ways to crate the committee with 3 boys and 7 girls. So the probability p is

$$p = \frac{\binom{15}{2}\binom{30}{7}}{\binom{45}{10}} \approx 0.2904.$$

□

Note. A similar analysis as that given in Example 1.8.8 can be used to calculate the probability of being dealt certain hands of cards. See Example 1.8.9 and Exercise 1.8.10.

Example. The Tennis Tournament. Quoting from pages 29 and 30: "We shall now present a difficult problem that has a simple and elegant solution. Suppose that n tennis players are entered in a tournament. In the first round the players are paired one against another at random. The loser in each pair is eliminated from the tournament, and the winner is each pair continues into the second round. If the number of players n is odd, then one player is chosen at random before the pairings are made for the first round, and [s]he automatically continues into the second round. All the players in the second round are then paired at random. Again, the loser in each pair is eliminated, and the winner in each pair continues into the third round. If the number of players in the second round is odd, then one of these players is chosen at random before the others are paired, and [s]he automatically continues into the third round. The tournament continues in this way until only two players remain in the final round. They then play against each other, and the winner of this match is the winner of the tournament. We shall assume that all n players have equal ability, and we shall determine the probability p that two specific players A and B will play against each other at any time during the tournament."

Since $n-1$ players must be eliminated then $n-1$ matches must be played. Due to the randomness in terms of being assigned to play a match and the randomness of a victory, the probability of particular players A and B together in a match is just $1/\binom{n}{2}$, since there are $\binom{n}{2}$ possible total matches. Since $n-1$ matches are played, the total probability that A and B play each other in the tournament is

$$p = \frac{n-1}{\binom{n}{2}} = \frac{n-1}{n(n-1)/2} = \frac{2}{n}. \qquad \square$$

Revised: 7/18/2019