Section 1.9. Multinomial Coefficients

Note. Consider a set of n distinct elements that fall into k different groups where $k \geq 2$) such that the jth group contains n_j elements (where $1 \leq j \leq k$) so that $n_1 + n_2 + \cdots + n_k = n$. We want to count the number of ways the n elements can be divided into the k groups.

Note. We solve this counting problem by first observing that the are $\binom{n}{n_1}$ ways to assign elements to the first group. Then there are $\binom{n-n_1}{n_2}$ ways to assign elements to the second group. Then there are $\binom{n-n_1-n_2}{n_3}$ ways to assign elements to the third group, and so forth. By the multiplication rule, the number of ways to assign all elements to the k groups is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-2}}{n_{k-1}}\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

(notice that the last term here is 1). In terms of factorials this equals

$$\frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{\underbrace{(n-n_1-n_2-\cdots-n_k)!}_{0!=1}n_k!} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

Definition 1.9.1. The *multinomial coefficients* for n elements in k categories of sizes $n_1, n_2, \ldots, n_k$ is $\dfrac{n!}{n_1!n_2!\cdots n_k!}$.

Theorem 1.9.1. Theorem Multinomial Theorem. For any real number $x_1, x_2, \ldots, x_k$ and $n \in \mathbb{N}$ we have

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!n_2!\cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Note. We can prove the Multinomial Theorem using mathematical induction. For $k = 2$, the Multinomial Theorem reduces to the Binomial Theorem.

Example 1.9.2. Choosing Committees. Suppose that 20 members of an organization are to be divided into three committees A, B, and C. Committees A and B will have 8 members and committee C will have 4 members. The number of possible committees is given by the multinomial coefficient with $n = 10$, $n_1 = n_2 = 8$, and $n_3 = 4$: $\frac{20!}{8!8!4!} = 62,355,150$. □

Example 1.9.4. Playing Cards. A standard deck of 52 cars contains 13 hearts. Four players are randomly dealt 13 cards. We calculate the probability p that player A gets 6 hearts, player B gets 4 hearts, player C gets 2 hearts, and player D gets 1 heart. With $n = 52$, $n_1 = n_2 = n_3 = n_4 = 13$ we have from the multinomial coefficient that there are

$$N = \frac{n!}{n_1!n_2!n_3!n_4!} = \frac{52!}{(13!)^4}$$

ways to deal the cards. As a second application of the binomial coefficients, we consider how the 13 hearts can be distributed amount the four players. We now take $n = 13$, $n_1 = 6$, $n_2 = 4$, $n_3 = 2$, and $n_1 = 1$ and see that there are $\frac{13!}{6!4!2!1!}$ ways to distribute the hearts. Finally, the remaining 39 non-hearts must be distributed among the four players. To count the number of ways this can be done, we take $n = 39$, $n_1 = 7$, $n_2 = 9$, $n_3 = 11$, and $n_4 = 12$ so that there are $\frac{39!}{7!9!11!12!}$ ways to do this. Hence, by the multiplication rule, there are

$$M = \frac{13!}{6!4!2!1!}\frac{39!}{7!9!11!12!}.$$

The desired probability is then

$$p = \frac{M}{N} = \frac{13!}{6!4!2!1!} \frac{39!}{7!9!11!12!} \Bigg/ \frac{52!}{(13!)^4} \approx 0.00196.$$

Alternatively, we can restrict our attention to just the location of the hearts. The 13 hearts can be dealt out in $\binom{52}{13}$ different ways. Player A can get 6 hearts in the 13 dealt to her/him in $\binom{13}{6}$ ways. Similarly, players B, C, and D can get their desired number of hearts in $\binom{13}{4}$, $\binom{13}{2}$, $\binom{13}{1}$ ways. So we also have:

$$p = \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} \approx 0.00196.$$

$\square$

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