Section 2.2. Independent Events

Note. Informally, two events are independent of one another if the occurrence of one event has no influence on the occurrence of the other.

Definition 2.2.1. Two events A and B are *independent* if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

Note. If $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$ are independent events then $\Pr(A \cap B) = \Pr(A)\Pr(B|A) = \Pr(B)\Pr(B|A) = \Pr(A)\Pr(B)$. Therefore, $\Pr(A) = \Pr(A|B)$ and $\Pr(B) = \Pr(B|A)$, justifying our informal idea that the occurrence of one event has no influence on the occurrence of the other event.

Exercise 2.2.2. Suppose events A and B are independent. Prove that events A^c and B^c are also independent.

Example 2.2.3. Rolling a Die. Suppose that a balanced die is rolled. Let A be the event that an even number is obtained, and let B be the event that one of the numbers 1, 2, 3, or 4 is obtained. Then $A \cap B$ is the vent that one of the numbers 2 or 4 is obtained. So $\Pr(A) = 3/6 = 1/2$, $\Pr(B) = 4/6 = 2/3$, and $\Pr(A \cap B) = 2/6 = 1/3$. So

$$\Pr(A \cap B) = \frac{1}{3} = \frac{1}{2}\frac{2}{3} = \Pr(A)\Pr(B)$$

and hence, by definition, A and B are independent. □

Theorem 2.2.1. If two events A and B are independent, then the events A and B^c are also independent.

Definition 2.2.2. Events $A_1, A_2, \ldots, A_k$ are *independent* if for every subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_j}\}$ of $\{A_1, A_2, \ldots, A_k\}$ we have

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}) = \Pr(A_{i_1})\Pr(A_{i_2}) \cdots \Pr(A_{i_j}).$$

Example 2.2.4. Pairwise Independence. Consider an experiment in which the sample space is $\{s_1, s_2, s_3, s_4\}$ where the probability of each outcome is 1/4. Let $A = \{s_1, s_2\}$, $B = \{s_1, s_3\}$, and $C = \{s_1, s_4\}$. Then $A \cap B = A \cap C = C \cap C = A \cap B \cap C = \{s_1\}$. Hence $\Pr(A) = \Pr(B) = \Pr(C) = 2/4 = 1/2$ and $\Pr(A \cap B) = \Pr(A \cap C) = \Pr(B \cap C) = \Pr(A \cap B \cap C) = 1/4$. So A, B, C are pairwise independent, but $1/8 = \Pr(A)\Pr(B)\Pr(C) \neq \Pr(A \cap B \cap C) = 1/4$, so events A, B, C are NOT independent. $\square$

Examples 2.2.5 and 2.2.6. Suppose that a machine produces a defective item with probability p (where $0 < p < 1$) and produces a nondefective item with probability $q = 1 - p$. Suppose it produces 6 items. We want to find the probability that 2 of the items are defective. If we ordered the items, the probability that the first 2 are defective and the last 4 are not (since those outcomes are independent) is p^2q^4. But we are not interested in order, so we count the number of ways 6 items can have 2 defective which is $\binom{6}{2} = \binom{6}{4}$. So the probability of 2 defective times in a set of 6 is $\binom{6}{2}p^2q^4$. Notice that the probability of 6 nondefectives is similarly $\binom{6}{0}p^0q^6 = q^6$, so the probability of at least one defective item in a group of 6 is $1 - q^6$.

Example 2.2.7. Tossing a Coin Until a Head Appears. Suppose that a fair coin is tossed until a head appears for the first time and assume the tosses are independent. Let p_n be the probability that exactly n tosses are required. So we are interested in the event of obtaining $n-1$ tails in succession and then obtaining a head on the next toss. So $p_n = (1/2)^n$. Notice that the probability that a head will be eventually obtained is $\displaystyle\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. So the probability that a head never is obtained is 0. This is an example of an event with probability 0, but the event is not impossible. This is a common occurrence in an infinite sample space.

Note. The proof of the following is to be given in Exercise 2.2.21.

Theorem 2.2.2. Let $A_1, A_2, \ldots, A_k$ be events such that $\Pr(A_1 \cap A_2 \cap \cdots \cap A_k) > 0$. Then $A_1, A_2, \ldots, A_k$ are independent if and only if, for every two disjoint subsets $\{i_1, i_2, \ldots, i_m\}$ and $\{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, k\}$ we have

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m} | A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_\ell}) = \Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}).$$

Note. The proof of the following is to be given in Exercise 2.2.24.

Theorem 2.2.3. Let $n > 1$ and let $A_1, A_2, \ldots, A_n$ be events that are mutually exclusive. The events are also mutually independent if and only if all the events except possibly one of them has probability 0.

Definition 2.2.3. We say that events $A_1, A_2, \ldots, A_k$ are *conditionally independent given* B if, for every subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_j}$ of j events (where $j \in \{1, 2, \ldots, k\}$), we have

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}|B) = \Pr(A_{i_1}|B)\Pr A_{i_2}|B) \cdots \Pr(A_{i_j}|B).$$

Note. The following is to be proved in Exercise 2.2.22.

Theorem 2.2.4. Suppose that A_1, A_2, and B are events such that $\Pr(A_1 \cap B) \neq 0$. Then A_1 and A_2 are conditionally independent given B if and only if $\Pr(A_2|A_1 \cap B) = \Pr(A_2|B)$.

Note. We now consider the **Collector's Problem**. Consider a collection of baseball cards. For this story, each package of cards contains a single card (which cannot be seen until the package is opened). A complete collection of cards consists of r cards. If someone buys n cards (where $n \geq r$) then we want the probability p that the person will get a complete set of r different pictures. For $i = 1, 2, \ldots, r$ let A_i be the event that a card of player i is mussing from all n packages. Then $\cup_{i=1}^r A_i$ is the event that at least one player's card is missing (think of union as "or"). We'll use Theorem 1.10.2 to find $\Pr\left(\cup_{i=1}^r A_i\right)$. Since there are r different cards then the probability that a particular package does not contain player i is $(r-1)/r$. So for n packages, the probability that all do not contain player i is $(r-1)/r)^n$; that is, $\Pr(A_i) = ((r-1)/r)^n$ for any $i = 1, 2, \ldots, r$. For $i \neq j$ the probability that neither the card for player i not the card for player j is in a particular package is

$(r-2)/r$. So for n packages, the probability that all do not contain players i and j is $(r-2)/r)^n$; that is, $\Pr(A_i \cap A_j) = ((r-2)/r)^n$. Similarly, for distinct i, j, k we have $\Pr(A_i \cap A_j \cap A_k) = ((r-3)/r)^n$. Also, for distinct events $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ we have $\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = ((r-m)/r)^n$ for $m = 1, 2, \ldots, r$. Now there are $\binom{r}{m}$ ways to choose the indices $i_1, i_2, \ldots, i_m$ so the probability that all n packages are missing m players is $\binom{r}{m}((r-m)/r)^n$. So by Theorem 1.10.2 we have

$$\Pr\left(\cup_{i=1}^r A_i\right) = \binom{r}{1}\left(\frac{r-1}{r}\right)^n - \binom{r}{2}\left(\frac{r-2}{r}\right)^n + \binom{r}{3}\left(\frac{r-3}{r}\right)^n + \cdots$$

$$+(-1)^r\binom{r}{r-1}\left(\frac{1}{r}\right)^n + 0 = \sum_{j=1}^{r-1}(-1)^{j+1}\binom{r}{j}\left(\frac{r-j}{r}\right)^n.$$

The desired probability p of obtaining a complete set of all r players is the probability of the event $\left(\cup_{i=1}^r A_i\right)^c$ and so

$$p = 1 - \Pr\left(\cup_{i=1}^r A_i\right) = \sum_{j=0}^{r-1}(-1)^j\binom{r}{j}\left(\frac{r-j}{r}\right)^n.$$

□

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