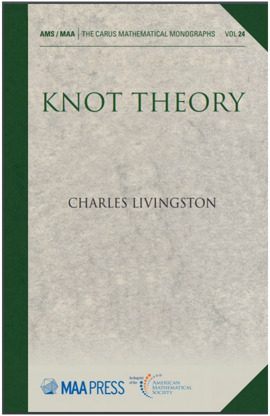
Introduction to Knot Theory

Chapter 3. Combinatorial Techniques

3.2. Colorings—Proofs of Theorems



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So we need to consider 6 cases. We present one case here, Reidemeister move 2b, and leave the other cases to the exercises. Suppose the Reidemeister move 2b is performed on a colored knot diagram. This requires us to consider two subcases. As illustrated in Figure 3.5(a), we consider a situation where 2 colors (and hence 3 colors by part (2) of the definition of colorable) are used in the arc colorings.

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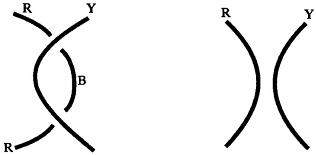

(*a*) *Figure 3.5* (*b*)

Figure 3.5(b) shows how the arcs are modified and how the arcs can be colored is such a way as to leave the rest of the diagram arc colors unchanged, hence preserving the colorability through Reidemeister move 2b in this subcase.

In the second subcase, we consider that all arcs (as configured in Figure 3.5(a)) are of the same color. We simply color the resulting arcs (as configured in Figure 3.5(b)) that color and this leaves the rest of the diagram arc colors unchanged, hence preserving the colorability through Reidemeister mover 2b in this subcase, and so in general.

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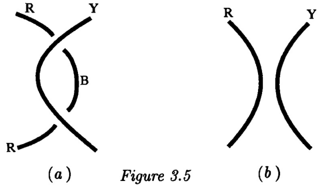

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Theorem 3.2.2 (continued 2)

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Proof (continued). The case of Reidemeister move 1a is to be done in Exercise 3.2.4(a). The number of subcases of Reidemeister move 3a are to be enumerated in Exercise 3.2.4(b), and are to be checked in Exercise 3.2.4(c). The remaining 3 cases are to be done in Exercise 3.2.4(d). □