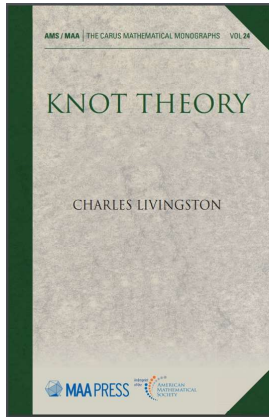


Introduction to Knot Theory

Chapter 4. Geometric Techniques

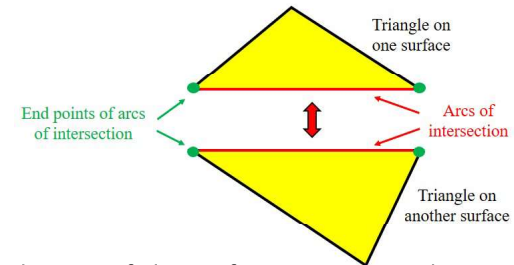
4.2. The Classification of Surfaces—Proofs of Theorems



Theorem 4.2.1

Theorem 4.2.1. If two surfaces intersect in a collection of arcs contained in their boundary, the Euler characteristic of the union is the sum of their individual Euler characteristics minus the number of (common) arcs of intersection.

Proof. By definition, the Euler characteristic of a surface S is $\chi(S) = F - E + V$. First, suppose that each arc of intersection is a single edge of a triangle on each surface:



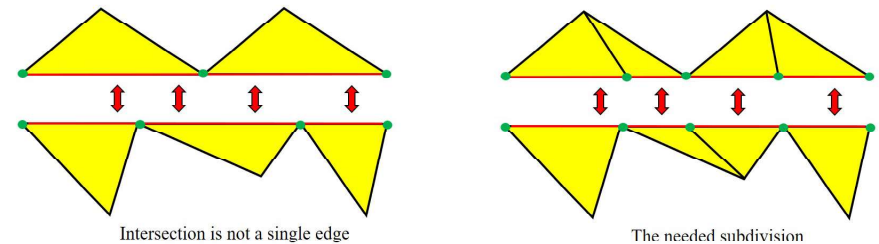
Then the triangulations of the surfaces piece together to give a triangulation of the union.

Theorem 4.2.1 (continued 1)

Proof (continued). Each edge in the intersection of two surfaces in a shared boundary count in the Euler characteristic of both surfaces in the term “ $-E$.” So in the computation of the Euler characteristic of the union, we must add 1 to account for the net loss of one edge in the union. Similarly, we have a net loss of two vertices in the union for each edge in the common boundary and, because of the computation of the Euler characteristic in the term $+V$, we must subtract 2 to account for this loss. So in this case, the Euler characteristic of the union is the sum of the Euler characteristics adjusted by $(1 - 2) = -1$ for each arc (i.e. each edge of a triangle in the boundary) in the common boundary. That is, in the case that each arc of intersection is a single edge of a triangle, then the Euler characteristic of the union is the sum of the Euler characteristics minus the number of arcs of intersection, as claimed.

Theorem 4.2.1 (continued 2)

Proof (continued).

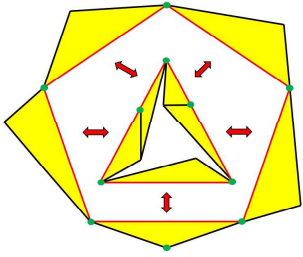


If an arc is not a single edge of a triangle, then we can subdivide the triangles as given here. Now the arc of intersection consists of one more vertex than edge and, as in the first case, the Euler characteristic of the union is the sum of the Euler characteristics adjusted by -1 for each arc in the intersection, as claimed. \square

Theorem 4.2.4

Theorem 4.2.4. If two surfaces intersect in a collection of circles contained in the boundary of each, the Euler characteristic of their union is the sum of their Euler characteristics.

Proof. Since subdivision does not change the Euler characteristic, we subdivide as need until both the shared boundary of two surfaces have the same number of edges:



When we then union together two such surfaces, we lose the same number of edges and vertices, so that the Euler characteristic $\chi = F - E + V$ of the union is the sum of the Euler characteristics. The general result then holds by induction. \square