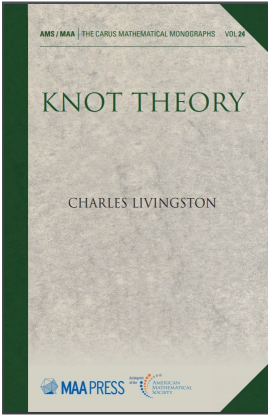
# Introduction to Knot Theory

## Chapter 4. Geometric Techniques

### 4.2. The Classification of Surfaces—Proofs of Theorems



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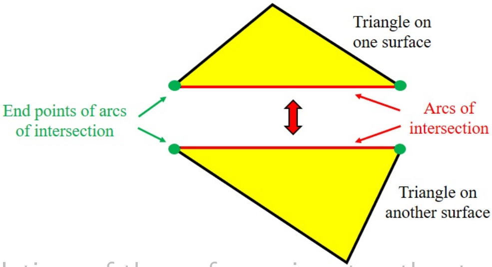


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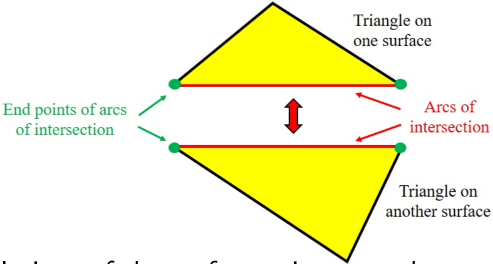


Then the triangulations of the surfaces piece together to give a triangulation of the union.

# Theorem 4.2.1 (continued 1)

**Proof (continued).** Each edge in the intersection of two surfaces in a shared boundary count in the Euler characteristic of both surfaces in the term "$-E$." So in the computation of the Euler characteristic of the union, we must add 1 to account for the net loss of one edge in the union. Similarly, we have a net loss of two vertices in the union for each edge in the common boundary and, because of the computation of the Euler characteristic in the term $+V$, we must subtract 2 to account for this loss. So in this case, the Euler characteristic of the union is the sum of the Euler characteristics adjusted by $(1-2) = -1$ for each arc (i.e. each edge of a triangle in the boundary) in the common boundary.

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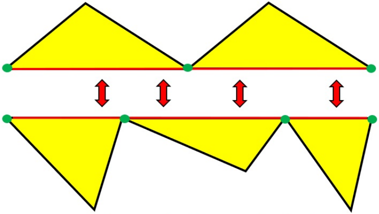
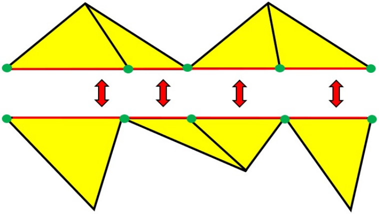


Intersection is not a single edge



The needed subdivision

If an arc is not a single edge of a triangle, then we can subdivide the triangles as given here. Now the arc of intersection consists of one more vertex than edge and, as in the first case, the Euler characteristic of the union is the sum of the Euler characteristics adjusted by $-1$ for each arc in the intersection, as claimed. □

# Theorem 4.2.4

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**Proof.** Since subdivision does not change the Euler characteristic, we subdivide as need until both the shared boundary of two surfaces have the same number of edges:

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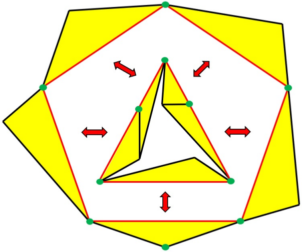

When we then union together two such surfaces, we loose the same number of edges and vertices, so that the Euler characteristic $\chi = F - E + V$ of the union is the sum of the Euler characteristics. The general result then holds by induction. □

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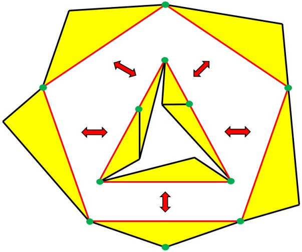

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