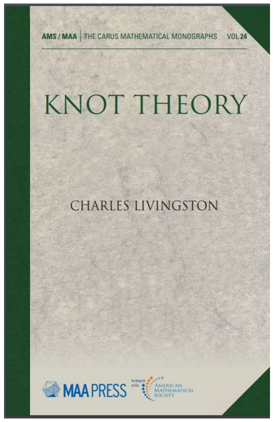
Introduction to Knot Theory

Chapter 4. Geometric Techniques

4.3. Seifert Surfaces and the Genus of a Knot—Proofs of Theorems



Theorem 4.3.7

Theorem 4.3.7. Every knot is the boundary of an orientable surface.

"Proof." We argue geometrically and somewhat informally. We start with an oriented diagram for the knot. Pick some point on an arc of the oriented diagram and trace around the diagram in the direction of the orientation. When a crossing is encountered, change arcs in such a way that the tracing continued in the direction of the knot, if possible. If this is not possible (which can occur when you return to a crossing; this will result in completing a directed cycle in the knot diagram), then go to a crossing for which you can continue tracing in the direction of the orientation. This procedure (we claim) leads to a collection of directed cycles (or "circles" as Livingston calls them); these are called *Seifert circles*. An illustration of this is given in Figure 4.12 below. The Seifert circles will next be used to construct an orientable surface.

Theorem 4.3.7 (continued 1)

"Proof (continued)."

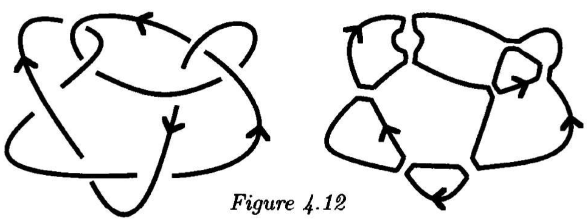
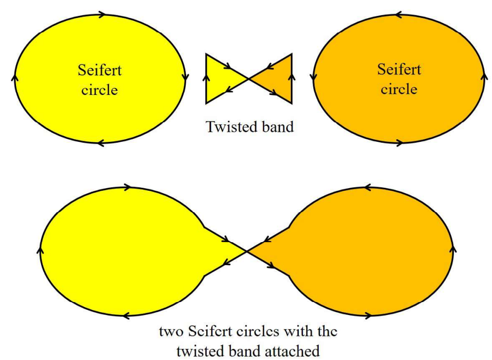


Figure 4.12

Create a disk for each Seifert circle with the circle as a boundary. Consider all circle as lying in a plane, but if some circle(s) lie inside other circles (as in the upper right of Figure 4.12), then lift the inner disk(s) above the outer disks, according to the nesting. To complete the Seifert surface, we connect the disks with twisted bands.

Theorem 4.3.7 (continued 2)

"Proof (continued)."



Connect disks along arcs corresponding to crossings by inserting twisted bands in such a way as to preserve the orientation of the knot diagram. The bands have been inserted in such a way that the original knot is the boundary of this surface

Theorem 4.3.7 (continued 3)

Theorem 4.3.7. Every knot is the boundary of an orientable surface.

"Proof (continued)." The surface is orientable, as is to be shown in Exercise 4.3.3. Therefore the surface as a Seifert surface of the original knot. □

Applying Seifert's algorithm to the oriented knot diagram of Figure 4.12 ives the following Seifert surface.

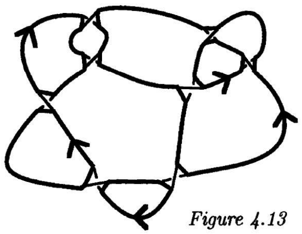


Figure 4.13