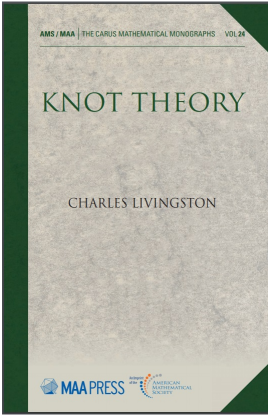
Introduction to Knot Theory

Chapter 4. Geometric Techniques

4.3. Seifert Surfaces and the Genus of a Knot—Proofs of Theorems



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Theorem 4.3.7 (continued 1)

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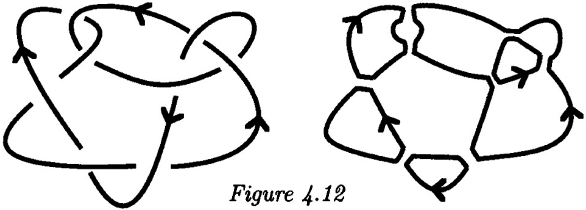
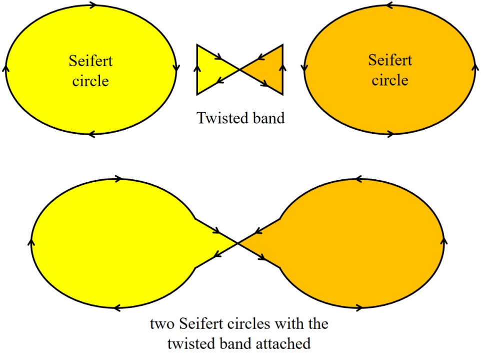


Figure 4.12

Create a disk for each Seifert circle with the circle as a boundary. Consider all circle as lying in a plane, but if some circle(s) lie inside other circles (as in the upper right of Figure 4.12), then lift the inner disk(s) above the outer disks, according to the nesting. To complete the Seifert surface, we connect the disks with twisted bands.

Theorem 4.3.7 (continued 2)

"Proof (continued)."



Connect disks along arcs corresponding to crossings by inserting twisted bands in such a way as to preserve the orientation of the knot diagram. The bands have been inserted in such a way that the original knot is the boundary of this surface

Theorem 4.3.7 (continued 3)

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"Proof (continued)." The surface is orientable, as is to be shown in Exercise 4.3.3. Therefore the surface as a Seifert surface of the original knot. □

Applying Seifert's algorithm to the oriented knot diagram of Figure 4.12 ives the following Seifert surface.

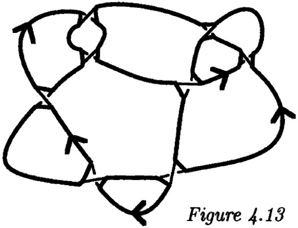

Figure 4.13