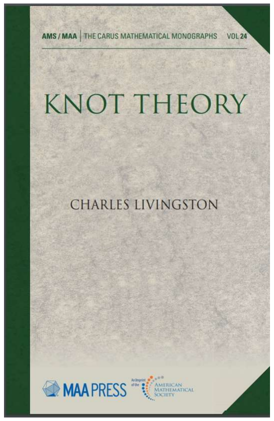
Introduction to Knot Theory

Chapter 4. Geometric Techniques

4.4. Surgery on Surfaces—Proofs of Theorems



Theorem 4.4.8

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Proof. In Example 4.2.1, we observed that (based on Theorem 4.2.1) if several bands are attached to a disk, then the Euler characteristic of the surface that results is $1 - \#\text{bands}$. So the Euler characteristic of a disk is 1 and of an annulus is 0 (think of an annulus as a disk with one band attached). In the first step of surgery, an annulus is removed from the surface. By Theorem 4.2.4, when two surfaces intersect in a collection of circles, the Euler characteristic of the union is the sum of the Euler characteristics; hence by removing an annulus the Euler characteristic remains the same, and by adding two disks increases the Euler characteristic by 2.

Theorem 4.4.8 (continued 1)

Proof (continued). If surgery results in a single new surface F', then the genus of that surface is given by $(2 - \chi(S) - B)/2$ (where B is the number of boundary components on the surface). A change in the Euler characteristic $\chi(S)$ by 2 implies a change in the genus by -1 (notice that B is the same before and after surgery). That is, $\text{genus}(F') = \text{genus}(F) - 1$, as claimed.

We now consider the case where the boundary of the disk is separating, so that surface F is separated into two surfaces F' and F'' by the surgery. The argument above concerning the Euler characteristic still holds and we have $\chi(F) + 2 = \chi(F') + \chi(F'')$. Let B, B', and B'' be the number of boundary components of F, F', and F'', respectively. Since surgery introduces no new boundary components nor does it delete any boundary components, so $B = B' + B''$.

Theorem 4.4.8 (continued 2)

Theorem 4.4.8. If surgery on a connected orientable surface F results in a connected surface F' then $\text{genus}(F') = \text{genus}(F) - 1$. If surgery results in a surface with two components, F' and F'', then $\text{genus}(F) = \text{genus}(F') + \text{genus}(F'')$.

Proof (continued). So from the definition of genus we have:

$$\begin{aligned}\text{genus}(F') + \text{genus}(F'') &= \frac{2 - \chi(F') - B'}{2} + \frac{2 - \chi(F'') - B''}{2} \\ &= \frac{4 - \chi(F') - \chi(F'') - B}{2} = \frac{4 - (\chi(F) + 2) - B}{2} \\ &= \frac{2 - \chi(F) - B}{2} = \text{genus}(F),\end{aligned}$$

as claimed. □