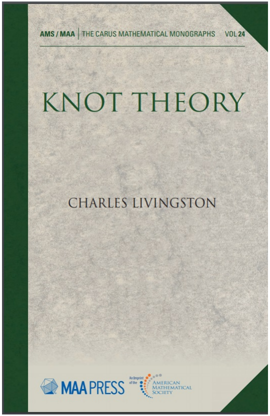
Introduction to Knot Theory

Chapter 4. Geometric Techniques

4.4. Surgery on Surfaces—Proofs of Theorems



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$\text{genus}(F) = \text{genus}(F') + \text{genus}(F'')$.

Proof. In Example 4.2.1, we observed that (based on Theorem 4.2.1) if several bands are attached to a disk, then the Euler characteristic of the surface that results is $1 - \#\text{bands}$. So the Euler characteristic of a disk is 1 and of an annulus is 0 (think of an annulus as a disk with one band attached).

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Theorem 4.4.8 (continued 1)

Proof (continued). If surgery results in a single new surface F', then the genus of that surface is given by $(2 - \chi(S) - B)/2$ (where B is the number of boundary components on the surface). A change in the Euler characteristic $\chi(S)$ by 2 implies a change in the genus by -1 (notice that B is the same before and after surgery). That is, $\text{genus}(F') = \text{genus}(F) - 1$, as claimed.

We now consider the case where the boundary of the disk is separating, so that surface F is separated into two surfaces F' and F'' by the surgery. The argument above concerning the Euler characteristic still holds and we have $\chi(F) + 2 = \chi(F') + \chi(F'')$. Let B, B', and B'' be the number of boundary components of F, F', and F'', respectively. Since surgery introduces no new boundary components nor does it delete any boundary components, so $B = B' + B''$.

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Theorem 4.4.8 (continued 2)

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Proof (continued). So from the definition of genus we have:

$$\text{genus}(F') + \text{genus}(F'') = \frac{2 - \chi(F') - B'}{2} + \frac{2 - \chi(F'') - B''}{2}$$

$$= \frac{4 - \chi(F') - \chi(F'') - B}{2} = \frac{4 - (\chi(F) + 2) - B}{2}$$

$$= \frac{2 - \chi(F) - B}{2} = \text{genus}(F),$$

as claimed. □