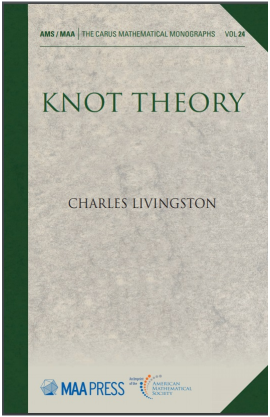
Introduction to Knot Theory

Chapter 4. Geometric Techniques

4.5. Connected Sums on Knots and Prime Decompositions—Proofs of Theorems



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If $K = K_1 \# K_2$ then $\text{genus}(K) = \text{genus}(K_1) + \text{genus}(K_2)$.

"Proof." First, we show that the genus of the connected sum is at least the sum of the genera. Let $K = K_1 \# K_2$ and let S be a separating sphere, as given in Figure 4.17. Let F be a minimal genus Seifert surface for the connected sum. We will modify F to a surface G of the same genus as F, but with G relating in a known way to the Seifert surfaces of K_1 and K_2.

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Now surface F intersects sphere S in a collection of arcs and circles on S (it could be that the surface is tangent to the sphere and so intersects the sphere at a single point, but we can then slightly perturb the surface without modifying its genus so that the intersection is a circle or the intersection no longer exists). In fact, any intersection of the surface with sphere S will be a "circle," except for the arc of intersection that runs between the two points on S that intersect $K = K_1 \# K_2$ (see Figure 4.17 again).

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Theorem 4.5.10 (continued 1)

"Proof (continued)." Consider a circle of intersection on sphere S that bounds a disk on S containing no points of intersection of surface F and S in its interior. Now perform surgery on surface F along the disk to construct a new surface bounded by knot K (see Figure 4.16 in Section 4.4 for a picture of the surgery; notice that this does not introduce any new boundary to the surface). If the new surface is connected then it is a Seifert surface for K which, by Theorem 4.4.8, has a genus one less than the genus of F, but this contradicts the minimality of the genus of the Seifert surface F. So the new surface must be disconnected surface with two components. One component must still have K as its boundary and this component is a Seifert surface for K, so discard the other component. By Theorem 4.4.8 again (for the case where surgery results in two components), the new Seifert surface has a genus less than or equal to that of F. The minimality of the genus of F implies that the new Seifert surface must have the same genus as F (and so the component that was discarded must have genus 0).

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"Proof (continued)." Also, the new Seifert surface has fewer circles of intersection with S, so we can repeat this surgery process until all circles of intersection have been removed and the only intersection of S with the resulting Seifert surface, which we denote as G, is the arc between the two points of intersection of knot K with sphere S. So G is a Seifert surface of knot $K = K_1 \# K_2$ since G is the bounded by K. In addition, we have that G is a minimal genus Seifert surface of K_1 "glued" along the arc on sphere S to a minimal genus Seifert surface of K_1 (the Seifert surface of one of the knots is contained in sphere S and the other Seifert surface is outside the sphere; see Figure 4.17 again). By Corollary 4.2.2, the genus of G is the sum of the genera of the Seifert surfaces for K_1 and K_2. That is $\text{genus}(K) = \text{genus}(K_1) + \text{genus}(K_2)$, as claimed. □

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Theorem 4.5.9. Prime Decomposition Theorem, Existence.
Every knot can be decomposed as the connected sum of nontrivial prime knots.

Proof. First, a trivial knot (the "'unknot") has genus 0 and no other knot has genus 0 (since it would then have a disk as it's Siefert surface and so is not knotted!). Notice that by Theorem 4.5.10 ("Additivity of Knot Genus") a knot of genus 1 must be prime since 1 is not the sum of any two positive integers.

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Every knot can be decomposed as the connected sum of nontrivial prime knots.

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We now give an inductive proof on the genus of knot K. First, a knot of genus 1 is a direct sum of nontrivial prime knots (the base case). Suppose every knot of genus greater than 1 and less than k can be decomposed as the connected sum of nontrivial prime knots (the induction hypothesis). Let K be a knot of genus k.

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$$K = P_1 \# P_2 \# \cdots \# P_\ell \# P_1' \# P_2' \# \cdots \# P_{\ell'}',$$

so K can be decomposed as the connected sum of nontrivial prime knots. Hence, by Mathematical Induction, every (nontrivial) knot can be decomposed as the connected sum of nontrivial prime knots, as claimed. □

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