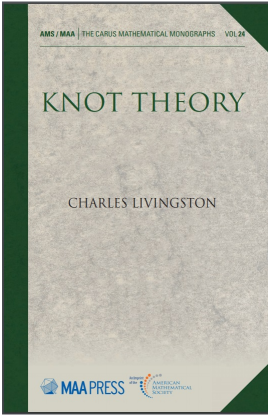
Introduction to Knot Theory

Chapter 5. Algebraic Techniques

5.2. Knots and Groups—Proofs of Theorems



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Theorem 5.2.1

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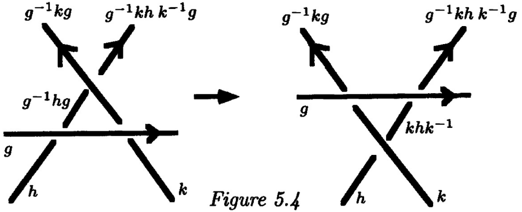

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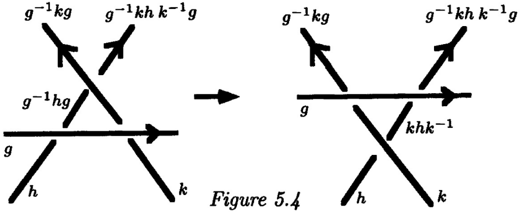

Figure 5.4

Theorem 5.2.1 (continued 1)

Partial Proof (continued).

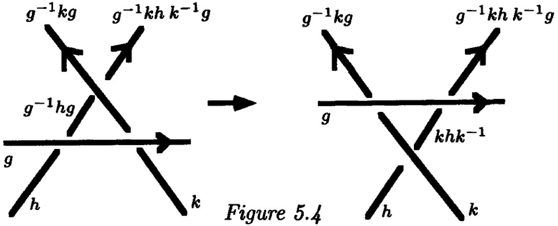

Figure 5.4

Notice that the labeling on the left is general, since it is done in terms of the three arbitrarily labeled arcs g, h and k. Based on this, we create the labeling given on the right. Before the Reidemeister move is performed, the upper crossing is left-handed and we have $(g^{-1}kg)(g^{-1}hg)(g^{-1}kg)^{-1} = g^{-1}kgg^{-1}hgg^{-1}k^{-1}g = g^{-1}khk^{-1}g$, as needed. The lower left crossing is right-handed and we have $(g)(g^{-1}hg)(g^{-1}) = gg^{-1}hgg^{-1} = h$, as needed. The lower right crossing is right-handed and we have $(g)(g^{-1}kg)(g^{-1}) = gg^{-1}kgg^{-1} = k$, as needed. So the condition of consistency is satisfied.

Theorem 5.2.1 (continued 1)

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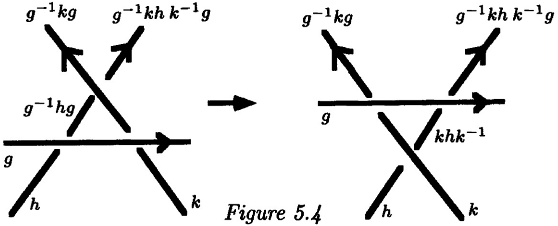

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Theorem 5.2.1 (continued 2)

Partial Proof (continued).

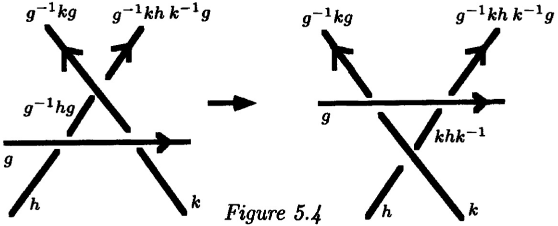

Figure 5.4

Notice that the labeling on the right only modifies the one "local" arc label; on the left we have label $g^{-1}hg$ and on the right we have khk^{-1}. After the Reidemeister move is performed, the upper left crossing is right-handed and we have $(g)(g^{-1}kg)(g^{-1}) = k$, as needed. The upper right crossing is right-handed and we have $(g)(g^{-1}khk^{-1}g)(g^{-1}) = gg^{-1}khk^{-1}gg^{-1} = khk^{-1}$, as needed. The lower crossing is left-handed and we have $(k)(h)(k^{-1}) = khk^{-1}$, as needed. So the condition of consistency is satisfied.

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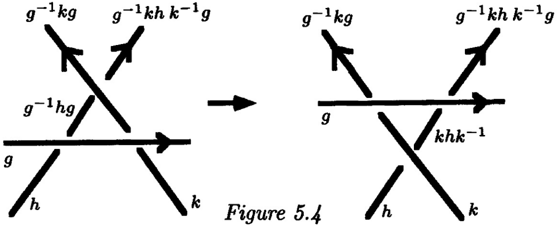

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Theorem 5.2.1 (continued 3)

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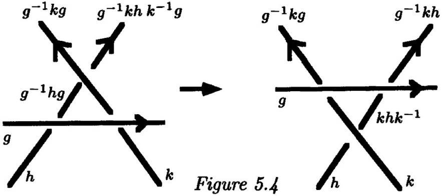

Figure 5.4

Now for the property of generation. Since group elements g, h, and k are used as labels before the Reidemeister move, then these group elements are in the generating set of the group from which the labels come. Set of arc labels $\{g, h, k\}$ generates all labels given in the diagram before the Reidemeister move is performed. Similarly, the labels g, h, and k are used after the Reidemeister move and again $\{g, h, k\}$ generates all labels in the diagram after the Reidemeister move is performed. So this Reidemeister move and the given labeling does not change the group generated by the arc labels. That is, the property of generation holds. Hence, for Reidemeister move 3a, the result holds. □

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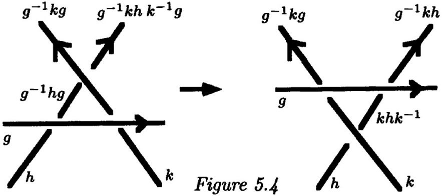

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