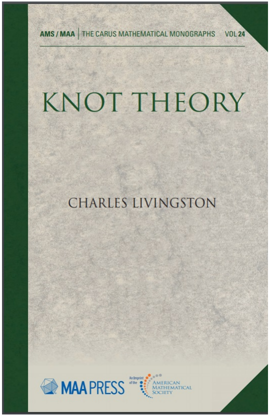
Introduction to Knot Theory

Chapter 5. Algebraic Techniques

5.3. Conjugation and the Labeling Theorem—Proofs of Theorems



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Theorem 5.3.A

Theorem 5.3.A. In the symmetric group S_n, two elements are conjugate if and only if they have the same cycle structure.

Proof. We base the proof on Exercise I.6.3 of Thomas W. Hungerford's *Algebra*, Graduate Texts in Mathematics #73, NY: Springer Verlag (1974), which state: If $\sigma = (i_1, i_2, \ldots, i_r) \in S_n$ and $\tau \in S_n$, then $\tau\sigma\tau^{-1}$ is the r-cycle $(\tau(i_1), \tau(i_2), \ldots, \tau(i_r))$. Let $\pi \in S_n$ and let

$$\pi = (a_1^1, a_2^1, \ldots, a_{n_1}^1)(a_1^2, a_2^2, \ldots, a_{n_2}^2) \cdots (a_1^r, a_2^r, \ldots, a_{n_r}^r)$$

be a unique (up to order of the factors) product of π as disjoint cycles, which exists by Hungerford's Theorem I.6.3.

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$$\pi = (a_1^1, a_2^1, \ldots, a_{n_1}^1)(a_1^2, a_2^2, \ldots, a_{n_2}^2) \cdots (a_1^r, a_2^r, \ldots, a_{n_r}^r)$$

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Theorem 5.3.A (continued 1)

Theorem 5.3.A. In the symmetric group S_n, two elements are conjugate if and only if they have the same cycle structure.

Proof (continued). Then for any $\tau \in S_n$ we have by conjugation τ that

$$\begin{aligned}
\tau\pi\tau^{-1} &= \tau(a_1^1, a_2^1, \ldots, a_{n_1}^1)\tau^{-1}\tau(a_1^2, a_2^2, \ldots, a_{n_2}^2)\tau^{-1} \cdots \\
&\quad \tau(a_1^r, a_2^r, \ldots, a_{n_r}^r)\tau^{-1} \\
&= (\tau(a_1^1), \tau(a_2^1), \ldots, \tau(a_{n_1}^1))(\tau(a_1^2), \tau(a_2^2), \ldots, \tau(a_{n_2}^2)) \cdots \\
&\quad (\tau(a_1^r), \tau(a_2^r), \ldots, \tau(a_{n_r}^r)) \text{ by Hungerford's Theorem I.6.3.}
\end{aligned}$$

Now τ is a bijection, so these cycles are also disjoint and hence the cycle type of $\tau\pi\tau^{-1}$ is the same as the cycle structure of π, as claimed.

Theorem 5.3.A (continued 2)

Theorem 5.3.A. In the symmetric group S_n, two elements are conjugate if and only if they have the same cycle structure.

Proof (continued). Now suppose π and ρ have the same cycle structure, say

$$\pi = (a_1^1, a_2^1, \ldots, a_{n_1}^1)(a_1^2, a_2^2, \ldots, a_{n_2}^2) \cdots (a_1^r, a_2^r, \ldots, a_{n_r}^r)$$

and

$$\rho = (b_1^1, b_2^1, \ldots, b_{n_1}^1)(b_1^2, b_2^2, \ldots, b_{n_2}^2) \cdots (b_1^r, b_2^r, \ldots, b_{n_r}^r).$$

Define τ mapping $\{1, 2, \ldots, n\}$ to itself defined as $\tau(a_i^j) = b_i^j$ for $1 \le i \le n_j$ and $1 \le j \le r$. Next, π and ρ have the same number of fixed points so we can extend τ to map the fixed points of π in a bijective way to the fixed points of ρ. Since the cycles in π are disjoint and the cycles in ρ are disjoint, then τ is a bijection. That is, τ is a permutation of $\{1, 2, \ldots, n\}$ and so $\tau \in S_n$.

Theorem 5.3.A (continued 3)

Theorem 5.3.A. In the symmetric group S_n, two elements are conjugate if and only if they have the same cycle structure.

Proof (continued). Now

$$\begin{aligned}\tau\pi\tau^{-1} &= (\tau(a_1^1), \tau(a_2^1), \ldots, \tau(a_{n_1}^1))(\tau(a_1^2), \tau(a_2^2), \ldots, \tau(a_{n_2}^2)) \cdots \\ &\quad (\tau(a_1^r), \tau(a_2^r), \ldots, \tau(a_{n_r}^r)) \text{ by Hungerford's Theorem I.6.3} \\ &= (b_1^1, b_2^1, \ldots, b_{n_1}^1)(b_1^2, b_2^2, \ldots, b_{n_2}^2) \cdots (b_1^r, b_2^r, \ldots, b_{n_r}^r) = \rho.\end{aligned}$$

That is, π and ρ are conjugates, as claimed. $\square$