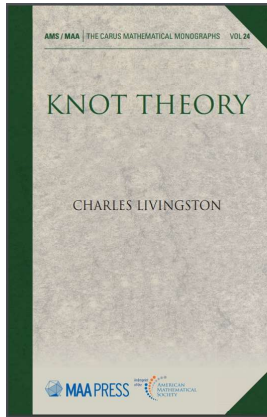


## Introduction to Knot Theory

### Chapter 6. Geometry, Algebra, and the Alexander Polynomial 6.2. Seifert Matrices and the Alexander Polynomial—Proofs of Theorems



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Corollary 6.2.2

### Corollary 6.2.2

**Corollary 6.2.2.** The Alexander polynomial of a knot  $K$  satisfies  $A_K(t) = t^{\pm i} A_K(t^{-1})$  for some  $i \in \mathbb{Z}$ .

**Proof.** This follows from properties of determinants. We recall some relevant properties from my online Linear Algebra (MATH 2010) notes on [4.2. The Determinant of a Square Matrix](#) (see Theorem 4.2.A. Properties of the Determinant): For square matrix  $A$  and scalar  $r$ ,  $\det(A) = \det(A^t)$  (the Transpose Property) and if a single row of  $A$  is multiplied by  $r$  to give matrix  $B$  then  $\det(B) = r\det(A)$  (The Scalar Multiplication Property). Notice that the second property implies that  $\det(rA) = r^n \det(A)$  where  $A$  is  $n \times n$ .

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Corollary 6.2.2

### Corollary 6.2.2 (continued)

**Corollary 6.2.2.** The Alexander polynomial of a knot  $K$  satisfies  $A_K(t) = t^{\pm i} A_K(t^{-1})$  for some  $i \in \mathbb{Z}$ .

**Proof.** We have that the Alexander polynomial satisfies

$$\begin{aligned} A_K(t) &= \det(V - tV^t) = \det((V - tV^t)^t) \text{ by the Transpose Property} \\ &= \det(V^t - tV) = \det(tV - V^t) \text{ by the Scalar Multiplication} \\ &\quad \text{Property with } r = -1 \text{ and } n = 2g \\ &= \det(t(V - t^{-1}V^t)) = t^{2g} \det(V - t^{-1}V^t) \text{ by the Scalar} \\ &\quad \text{Multiplication Property with } r = t \text{ and } n = 2g \\ &= t^{2g} A_K(t^{-1}). \end{aligned}$$

as claimed.  $\square$

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Corollary 6.2.4

### Corollary 6.2.4

**Corollary 6.2.4.** If  $V_1$  and  $V_2$  are Seifert matrices associated with the same knot, then the polynomials  $\det(V_1 - tV_1^t)$  and  $\det(V_2 - tV_2^t)$  differ by a multiple of  $\pm t^k$ .

**Proof.** We know that the  $V_1$  and  $V_2$  are  $S$ -equivalent by Theorem 6.2.3. So we consider the effect on  $\det(V_1 - tV_1^t)$  and  $\det(V_2 - tV_2^t)$  by the multiplication on the left by matrix  $M$  and on the right by matrix  $M^t$ , where  $M$  is as described in Note 6.2.A. Now  $M$  is a product of elementary matrices which correspond to adding a multiple of one row to another. The determinant of such an elementary matrix is the same as the identity matrix (by, say, The Row-Addition Property of “Theorem 4.2.A. Properties of the Determinant” in my Linear Algebra [MATH 2010] notes on [4.2. The Determinant of a Square Matrix](#)) and so is 1.

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## Corollary 6.2.4 (continued 1)

**Proof (continued).** Since the determinant of a product of matrices is the product of the determinants (by “Theorem 4.4. The Multiplicative Property” in the same online notes), then  $\det(M) = 1$  and  $\det(M^t) = 1$  (by The Transpose Property of Theorem 4.2.A in the online notes). So the band moves has the effect:

$$\det(M(V_1 - tV_1^t)M^t) = \det(M)\det(V_1 - tV_1^t)\det(M^t) = \det(V_1 - tV_1^t).$$

Now for stabilization. One step of stabilization changes Seifert matrix  $V_1$  to

$$V_1' = \begin{pmatrix} & & * & 0 \\ & V_1 & \vdots & \vdots \\ & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

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## Corollary 6.2.4 (continued 3)

**Proof (continued).**

$$\det(V_1' - t(V_1')^t) = (-1)^{(2g+1)+(2g+2)} \det \begin{pmatrix} & & * \\ & V_1 - tV_1^t & \vdots \\ & & * \\ 0 & \cdots & 0 & -t \end{pmatrix}$$

$$= (-1)(-t)(-1)^{(2g+1)+(2g+1)} \det(V_1 - tV_1^t) \text{ expanding along the last row} \\ = t \det(V_1 - tV_1^t).$$

So one step of stabilization affects  $\det(V_1 - tV_1^t)$  by a multiple of  $t$ .

So a sequence of band moves and stabilization will affect the determinant of  $V_1 - tV_1^t$  by some power of  $t$ . The claim not follows.  $\square$

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## Corollary 6.2.4 (continued 2)

**Proof (continued).** So we have

$$\det(V_1' - t(V_1')^t) = \det \left( \begin{pmatrix} & & * & 0 \\ & V_1 & \vdots & \vdots \\ & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - t \begin{pmatrix} & & * & 0 \\ & V_1^t & \vdots & \vdots \\ & & * & 0 \\ * & \cdots & * & * & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \right) \\ = \det \begin{pmatrix} & & * & 0 \\ & V_1 - tV_1^t & \vdots & \vdots \\ & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & -t & 0 \end{pmatrix}$$

We expand along the last column (and observe that we have a  $(2g+2) \times (2g+2)$  matrix).

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