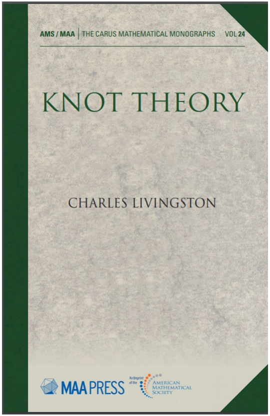
Introduction to Knot Theory

Chapter 6. Geometry, Algebra, and the Alexander Polynomial

6.2. Seifert Matrices and the Alexander Polynomial—Proofs of Theorems



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Proof. This follows from properties of determinants. We recall some relevant properties from my online Linear Algebra (MATH 2010) notes on 4.2. The Determinant of a Square Matrix (see Theorem 4.2.A. Properties of the Determinant): For square matrix A and scalar r, $\det(A) = \det(A^t)$ (the Transpose Property) and if a single row of A is multiplied by r to give matrix B then $\det(B) = r\det(A)$ (The Scalar Multiplication Property). Notice that the second property implies that $\det(rA) = r^n \det(A)$ where A is $n \times n$.

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Corollary 6.2.2 (continued)

Corollary 6.2.2. The Alexander polynomial of a knot K satisfies $A_K(t) = t^{\pm i} A_K(t^{-1})$ for some $i \in \mathbb{Z}$.

Proof. We have that the Alexander polynomial satisfies

$$\begin{aligned}
A_K(t) &= \det(V - tV^t) = \det((V - tV^t)^t) \text{ by the Transpose Property} \\
&= \det(V^t - tV) = \det(tV - V^t) \text{ by the Scalar Multiplication} \\
&\quad \text{Property with } r = -1 \text{ and } n = 2g \\
&= \det(t(V - t^{-1}V^t) = t^{2g}\det(V - v^{-1}V^t) \text{ by the Scalar} \\
&\quad \text{Multiplication Property with } r = t \text{ and } n = 2g \\
&= t^{2g} A + K(t^{-1}).
\end{aligned}$$

as claimed. □

Corollary 6.2.4

Corollary 6.2.4. If V_1 and V_2 are Seifert matrices associated with the same knot, then the polynomials $\det(V_1 - tV_1^t)$ and $\det(V_2 - tV_2^t)$ differ by a multiple of $\pm t^k$.

Proof. We know that the V_1 and V_2 are S-equivalent by Theorem 6.2.3 So we consider the effect on $\det(V_1 - tV_1^t)$ and $\det(V_2 - tV_2^t)$ by the multiplication on the left by matrix M and on the right by matrix M^t, where M is as described in Note 6.2.A. Now M is a product of elementary matrices which correspond to adding a multiple of one row to another. The determinant of such an elementary matrix is the same as the identity matrix (by, say, The Row-Addition Property of "Theorem 4.2.A. Properties of the Determinant" in my Linear Algebra [MATH 2010] notes on 4.2. The Determinant of a Square Matrix) and so is 1.

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Corollary 6.2.4 (continued 1)

Proof (continued). Since the determinant of a product of matrices is the product of the determinants (by "Theorem 4.4. The Multiplicative Property" in the same online notes), then $\det(M) = 1$ and $\det(M^t) = 1$ (by The Transpose Property of Theorem 4.2.A in the online notes). So the band moves has the effect:

$$\det(M(V_1 - tV_1^t)M^t) = \det(M)\det(V_1 - tV_1^t)\det(M^t) = \det(V_1 - tV_1^t).$$

Now for stabilization. One step of stabilization changes Seifert matrix V_1 to

$$V_1' = \begin{pmatrix} & & & * & 0 \\ & V_1 & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

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Corollary 6.2.4 (continued 2)

Proof (continued). So we have

$$\det(V_1' - t(V_1')^t) = \det\left(\begin{pmatrix} & & & * & 0 \\ & V_1 & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - t\begin{pmatrix} & & & * & 0 \\ & V_1^t & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & * & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} & & & * & 0 \\ & V_1 - tV_1^t & & \vdots & \vdots \\ & & & * & 0 \\ * & \cdots & * & * & 1 \\ 0 & \cdots & 0 & -t & 0 \end{pmatrix}$$

We expand along the last column (and observe that we have a $(2g+2) \times (2g+2)$ matrix).

Corollary 6.2.4 (continued 3)

Proof (continued).

$$\det(V_1' - t(V_1')^t) = (-1)^{(2g+1)+(2g+2)} \det \begin{pmatrix} & & & * \\ & V_1 - tV_1^t & & \vdots \\ & & & * \\ 0 & \cdots & 0 & -t \end{pmatrix}$$

$= (-1)(-t)(-1)^{(2g+1)+(2g+1)} \det(V_1 - tV_1^t)$ expanding along the last row

$$= t \det(V_1 - tV_1^t).$$

So one step of stabilization affects $\det(V_1 - tV_1^t)$ by a multiple of t.

So a sequence of band moves and stabilization will affect the determinant of $V_1 - tV_1^t$ by some power of t. The claim not follows. □

Corollary 6.2.4 (continued 3)

Proof (continued).

$$\det(V_1' - t(V_1')^t) = (-1)^{(2g+1)+(2g+2)} \det \begin{pmatrix} & V_1 - tV_1^t & & * \\ & & & \vdots \\ & & & * \\ 0 & \cdots & 0 & -t \end{pmatrix}$$

$$= (-1)(-t)(-1)^{(2g+1)+(2g+1)} \det(V_1 - tV_1^t) \text{ expanding along the last row}$$

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So a sequence of band moves and stabilization will affect the determinant of $V_1 - tV_1^t$ by some power of t. The claim not follows. □