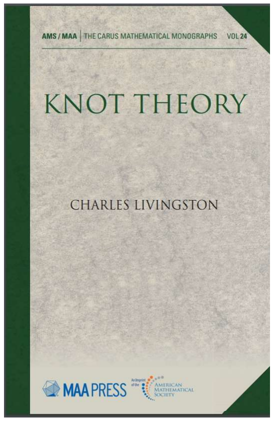
Introduction to Knot Theory

Chapter 8. Symmetries of Knots

8.2. Periodic Knots—Proofs of Theorems



Theorem 8.1

Theorem 8.1. If a knot diagram for K misses the origin, the corresponding q-fold covering link L has a single component if the linking number is relatively prime to q. More generally, the number of complements in L is the greatest common divisor of the linking number λ and q.

Proof. Neither changes in crossings nor deformations that do not cross the origin affect the linking number of the number of component in the covering link (deformations can increase the number of crossings of the ray, but for each new left-hand crossing added a right hand-crossing will be added and the linking number remains unchanged). Such deformations of a knot diagram determine periodic deformations of the covering link (called *lifts* of the deformation from the quotient knot to the periodic knot). Crossing changes do not affect orientations of crossings of the ray and so do not affect the linking number.

Theorem 8.1 (continued 1)

Proof (continued). Livingston states (see page 160): "By an appropriate sequence of crossing changes and deformations, the knot diagram can be transformed into one that runs monotonically around the axis. Crossing changes are used to eliminate any clasps that occur." Let's interpret this as given in Figure 8.9 (modified from Livingston's version). First, the knot on the left is modified with deformations (Reidemeister moves) to give the knot in the center. Then the crossings are "changed" in such a way as to eliminate the "clasp," but to modify the orientation such that the knot diagram runs clockwise (say) around the origin. Denote the new knot as K'.

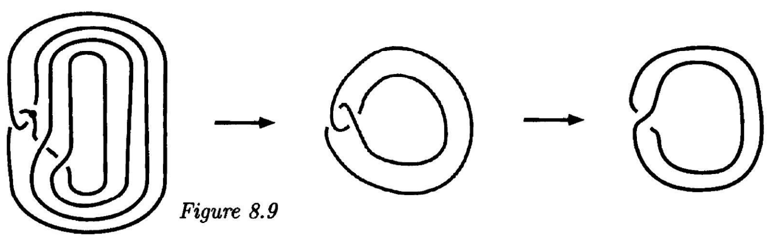


Figure 8.9

Theorem 8.1 (continued 2)

Proof (continued). With the knot K' oriented so that the knot diagram runs clockwise around the origin, pick a ray from the origin meeting in λ points, and label the points with integers from 1 to λ (from "outside the knot" to the origin). If we follow a particular labeled point around the knot (counterclockwise in the figure below), then we'll see that point move to a point with a (potentially) different label. So we can associate a permutation $\rho \in S_\lambda$ with the knot. In Figure 8.10, the permutation is $\rho = (1, 3, 4, 5, 2)$.

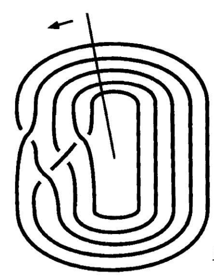


Figure 8.10

Theorem 8.1 (continued 3)

Theorem 8.1. If a knot diagram for K misses the origin, the corresponding q-fold covering link L has a single component if the linking number is relatively prime to q. More generally, the number of complements in L is the greatest common divisor of the linking number λ and q.

Proof (continued). In Figure 8.10, knot K' (right) is connected, so the permutation ρ is a λ-cycle. In general, K' would have one component for each cycle in a decomposition of ρ as a product of disjoint cycles (including 1-cycles); recall that every permutation of S_n can be written as a disjoint union of cycles (see my online notes on Introduction to Modern Algebra [MATH 4127/5127] on Section II.9. Orbits, Cycles, Alternating Groups; see Theorem 9.8).

Theorem 8.1 (continued 4)

Theorem 8.1. If a knot diagram for K misses the origin, the corresponding q-fold covering link L has a single component if the linking number is relatively prime to q. More generally, the number of complements in L is the greatest common divisor of the linking number λ and q.

Proof (continued. If we make a q-fold covering link L' based on K', then we get a permutation, say ρ', and it is "easily seen" from the construction that $\rho' = \rho^q$ (each time we "go around" link L' once, we have equivalently "gone around" K' q times). If q is relatively prime to λ, then the qth power of a λ-cycle is again a λ-cycle (a given cycle generates a cyclic subgroup of S_n and this claim concerning q relatively prime to λ follows from Theorem I.3.6 in my notes for Modern Algebra 1 [MATH 5410] on Section I.3. Cyclic Groups). "More generally," the qth power of a λ-cycle is the product of d disjoint λ/d cycles, where d is the greatest common divisor of a and λ (this follows from Theorem I.3.4 in the Modern Algebra 1 notes). □