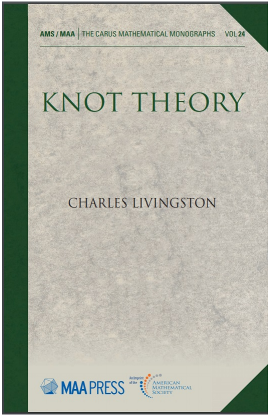
Introduction to Knot Theory

Chapter 8. Symmetries of Knots

8.2. Periodic Knots—Proofs of Theorems



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Theorem 8.1

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Proof. Neither changes in crossings nor deformations that do not cross the origin affect the linking number of the number of component in the covering link (deformations can increase the number of crossings of the ray, but for each new left-hand crossing added a right hand-crossing will be added and the linking number remains unchanged).

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Theorem 8.1 (continued 1)

Proof (continued). Livingston states (see page 160): "By an appropriate sequence of crossing changes and deformations, the knot diagram can be transformed into one that runs monotonically around the axis. Crossing changes are used to eliminate any clasps that occur." Let's interpret this as given in Figure 8.9 (modified from Livingston's version). First, the knot on the left is modified with deformations (Reidemeister moves) to give the knot in the center. Then the crossings are "changed" in such a way as to eliminate the "clasp," but to modify the orientation such that the knot diagram runs clockwise (say) around the origin. Denote the new knot as K'.

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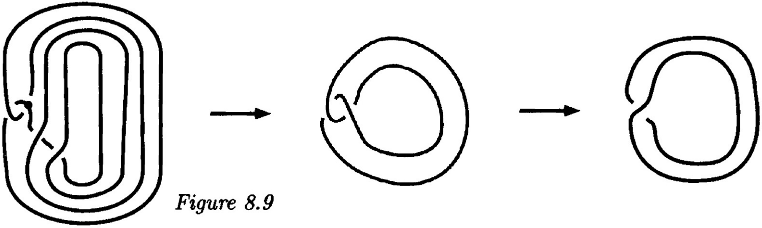

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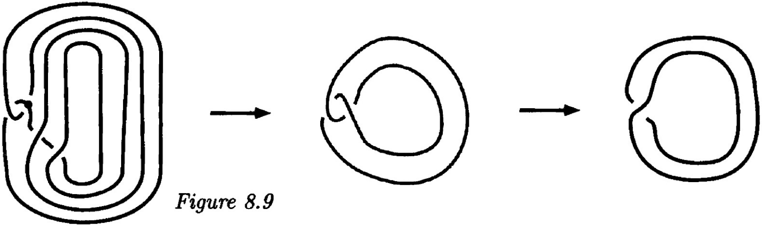

Figure 8.9

Theorem 8.1 (continued 2)

Proof (continued). With the knot K' oriented so that the knot diagram runs clockwise around the origin, pick a ray from the origin meeting in λ points, and label the points with integers from 1 to λ (from "outside the knot" to the origin). If we follow a particular labeled point around the knot (counterclockwise in the figure below), then we'll see that point move to a point with a (potentially) different label. So we can associate a permutation $\rho \in S_\lambda$ with the knot. In Figure 8.10, the permutation is $\rho = (1, 3, 4, 5, 2)$.

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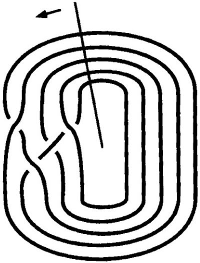

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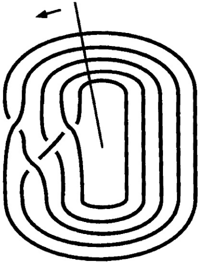

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Proof (continued). In Figure 8.10, knot K' (right) is connected, so the permutation ρ is a λ-cycle. In general, K' would have one component for each cycle in a decomposition of ρ as a product of disjoint cycles (including 1-cycles); recall that every permutation of S_n can be written as a disjoint union of cycles (see my online notes on Introduction to Modern Algebra [MATH 4127/5127] on Section II.9. Orbits, Cycles, Alternating Groups; see Theorem 9.8).

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