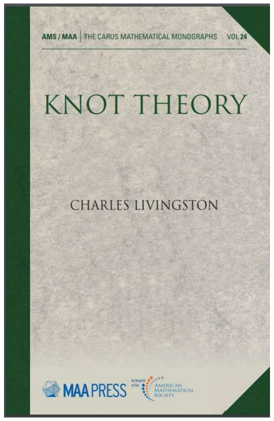
Introduction to Knot Theory

Chapter 8. Symmetries of Knots

8.4. Periodic Seifert Surfaces and Edmonds' Theorem—Proofs of Theorems



Theorem 8.4.4

Theorem 8.4.4. Riemann-Hurwitz Formula.
Let F be a genus g oriented surface which is equivariant with respect to a rotation about the z-axis of angle $(360/q)^\circ$, and let G be the quotient of F. If both F and G have one boundary component, then

$$\text{genus}(F) = q(\text{genus}(G)) + (q-1)(\Lambda - 1)/2,$$

where Λ is the number of points of intersection of F (or G) with the z-axis.

Proof. Recall from Section 4.2 ("Classification of Surfaces") that for polyhedral surface S which is triangulated with F triangles, with a total of E edges and V vertices in the triangulation, then the Euler characteristic is given by $\chi(S) = F - E + V$. So we consider triangulations of surfaces F and G. A triangulation of quotient surface G can be picked so that the intersection points of G with the z-axis are all vertices (this claim is based on geometric intuition). For this triangulation of G there is a corresponding triangulation of surface F.

Theorem 8.4.4 (continued 1)

Proof (continued). Since F is equivariant and admits a rotation through $(360/q)^\circ$, then each triangle in the triangulation of quotient surface G determines q triangles in the triangulation of surface F. Similarly, each edge of quotient surface G determines q edges on surface F. Also, the vertices of quotient G which are not on the z-axis lift to q vertices in F. But the Λ vertices of quotient G which are on the z-axis each lift to a single vertex in the triangulation of surface F.

Denote the number of triangles, edges, and vertices of the triangulation of surface F as t_F, e_F, and v_F, respectively. Similar define t_G, e_G, and v_G for surface G. By the argument above, we have $t_F = q\,t_G$, $e_F = q\,e_G$, and $v_F = q\,v_G - (q-1)\Lambda$. Now by definition, the genus of a connected orientable surface S is $\text{genus}(S) = (2 - \chi(S) - B)/2$ where B is the number of boundary components of the surface. Hence,

$$\chi(F) = t_F - e_F + v_F = q\,t_G - q\,e_G + q\,v_G - (q-1)\Lambda$$

and $\chi(G) = t_G - e_G + v_G$.

Theorem 8.4.4 (continued 2)

Proof (continued). Since surfaces F and G both have one boundary component by hypothesis, so

$$\begin{aligned}
\text{genus}(F) &= \frac{2 - \chi(F) - (1)}{2} = \frac{2 - (q\,t_G - q\,e_G + q\,v_G - (q-1)\Lambda) - 1}{2} \\
&= \frac{2 - q(t_G - e_G + v_G) + (q-1)\Lambda - 1}{2} \\
&= \frac{2 - q\,\chi(G) - (1)}{2} + \frac{(q-1)\Lambda}{2} \\
&= \frac{1}{2} - \frac{q\,\chi(G)}{2} + \left(\frac{q}{2} - \frac{q}{2}\right) + \frac{(q-1)\Lambda}{2} \\
&= \frac{2q - q\,\chi(G) - q}{2} - \frac{q}{2} + \frac{1}{2} + \frac{(q-1)\Lambda}{2} \\
&= q\frac{2 - \chi(G) - (1)}{2} + \frac{(q-1)\Lambda - (q-1)}{2} \\
&= q(\text{genus}(G)) + \frac{(q-1)(\Lambda - 1)}{2}, \text{ as claimed.} \quad \square
\end{aligned}$$