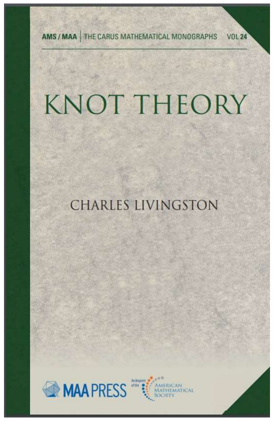
Introduction to Knot Theory

Chapter 8. Symmetries of Knots

8.5. Applications of the Murasugi and Edmonds Conditions—Proofs of Theorems



Corollary 8.5.7

Corollary 8.5.7. If a genus 1 knot K has period 3, then its Alexander polynomial satisfies $A_K(t) = \pm t^i(t^2 + 2t + 1) \pmod 3$.

Proof. With $g(K) = 1$ and $q = 3$, then Edmond's Condition (Corollary 8.4.6), $g(K) = q\, g_G + (q-1)(\Lambda - 1)/2$ implies that $1 = 3g_G + 2(\Lambda - 1)/2 = 3g_G + \Lambda - 1$ and so we must have $g_G = 0$ and $\Lambda = 2$. Now the linking number λ of the quotient knot satisfies $\lambda \leq \Lambda$, $\Lambda = \lambda \pmod 2$ and λ is relatively prime with $q = 3$ by Edmond's Conditions (Corollary 8.4.6), we we must have $\lambda = 2$. By Theorem 6.2.1, a genus 1 knot has an Alexander polynomial of degree at most 2, so with $p = q = 3$ and $\lambda = 2$ in the Murasugi Conditions (Theorem 8.3.2(2)), we must have $A_K(t) = \pm t^i (A_J(t))^q (1+t)^2 \pmod 3$, or $A_K(t) = \pm t^i(t^2 + 2t + 1) \pmod 3$, as claimed. Also notice that $A_J(t)$ must be a constant and hence of degree 0. □

Corollary 8.5.8

Corollary 8.5.8. If a genus 2 knot K has period 3, then its Alexander polynomial satisfies $A_K(t) = \pm t^i \pmod 3$.

Proof. We have $g(K) = 2$ and $q = 3$. By Exercise 8.4.2(a), Edmond's Conditions (Corollary 8.4.6) imply that the only possible values for the remaining parameters are $g_G = 0$ and $\lambda = 3$ (where g_G is the genus of the quotient knot J). Hence the quotient knot is trivial (its Seifert surface is a disk, since its genus is 0) and so has the trivial Alexander polynomial 1. By Edmond's Conditions (Corollary 8.4.6), we have $\Lambda \geq \lambda$ and $\Lambda = \lambda \pmod 2$, so the only possible values for λ are 1 or 3. Edmond's Conditions also implies that λ and $q = 3$ are relatively prime, so we must have $\lambda = 1$. Then by Murasugi's second condition (Theorem 8.3.2(2)), $A_k(t) = \pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^{q-1} \pmod p$, or $A_K(t) = \pm t^i (1)^3 (1)^2 \pmod 3$ or $A_K(t) = \pm t^i$, as claimed. □

Corollary 8.5.9

Corollary 8.5.9. If a nontrivial knot K is of period 5 and $g(K) \leq 3$, then the Alexander polynomial of K satisfies $A_K(t) = \pm t^i(t^4 - t^3 + t^2 - t + 1) \pmod 5$, and genus$(K) = 2$.

Proof. We have $q = 5$ and $g(K) \leq 3$. Edmond's Conditions (Corollary 8.4.6) imply that $g(K) = q\, g_G + (q-1)(\Lambda - 1)/2$, or $g(K) = 5g_G + 4(\Lambda - 1)$. There are no solutions for $g(K) = 1$ nor for $g(K) = 3$. For $g(K) = 2$, we have the only solution when $g_G = 0$ and $\Lambda = 2$. Edmond's Conditions also imply that $\Lambda \geq \lambda$, $\Lambda = \lambda \pmod 2$, and λ is relatively prime to $q = 5$. So we must have $\lambda = 2$. Since $g_G = 0$ is the genus of the quotient knot J, the the quotient knot J is trivial (since it bounds a genus 0 Seifert surface) and so the Alexander polynomial of J is trivial, $A_J(t) = 1$. Then by Murasugi's second condition (Theorem 8.3.2(2)), $A_k(t) = \pm t^i (A_J(t))^q (1 + t + t^2 + \cdots + t^{\lambda-1})^(q-1) \pmod p$, or $A_K(t) = \pm t^i (1)^5 (1+t)^4 \pmod 5$ or $A_K(t) = \pm t^i(t^4 + 4t^3 + 6t^2 + 4t + 1) \pmod 5$ or $A_K(t) = \pm t^i(t^4 - t^3 + t^2 - t + 1) \pmod 5$, as claimed. □