Introduction to Knot Theory

Chapter 8. Symmetries of Knots

8.5. Applications of the Murasugi and Edmonds Conditions—Proofs of Theorems

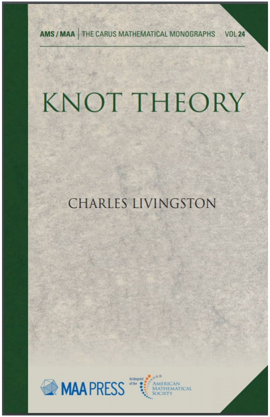

Table of contents

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Proof. With $g(K) = 1$ and $q = 3$, then Edmond's Condition (Corollary 8.4.6), $g(K) = q\, g_G + (q-1)(\Lambda - 1)/2$ implies that $1 = 3g_G + 2(\Lambda - 1)/2 = 3g_G + \Lambda - 1$ and so we must have $g_G = 0$ and $\Lambda = 2$. Now the linking number λ of the quotient knot satisfies $\lambda \leq \Lambda$, $\Lambda = \lambda \pmod 2$ and λ is relatively prime with $q = 3$ by Edmond's Conditions (Corollary 8.4.6), we we must have $\lambda = 2$.

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Corollary 8.5.8

Corollary 8.5.8. If a genus 2 knot K has period 3, then its Alexander polynomial satisfies $A_K(t) = \pm t^i \pmod 3$.

Proof. We have $g(K) = 2$ and $q = 3$. By Exercise 8.4.2(a), Edmond's Conditions (Corollary 8.4.6) imply that the only possible values for the remaining parameters are $g_G = 0$ and $\lambda = 3$ (where g_G is the genus of the quotient knot J). Hence the quotient knot is trivial (its Seifert surface is a disk, since its genus is 0) and so has the trivial Alexander polynomial 1.

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Proof. We have $q = 5$ and $g(K) \leq 3$. Edmond's Conditions (Corollary 8.4.6) imply that $g(K) = q\, g_G + (q-1)(\Lambda - 1)/2$, or $g(K) = 5g_G + 4(\Lambda - 1)$. There are no solutions for $g(K) = 1$ nor for $g(K) = 3$. For $g(K) = 2$, we have the only solution when $g_G = 0$ and $\Lambda = 2$. Edmond's Conditions also imply that $\Lambda \geq \lambda$, $\Lambda = \lambda$ (mod 2), and λ is relatively prime to $q = 5$. So we must have $\lambda = 2$.

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