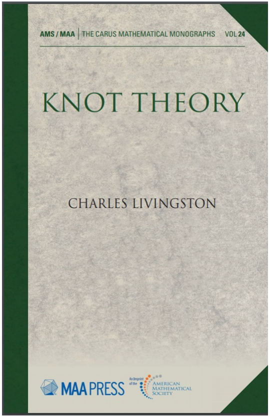
Introduction to Knot Theory

Chapter 9. High-Dimensional Knot Theory

9.4. Slice Knots—Proofs of Theorems



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Proof. The Alexander polynomial is $\det(V - tV^t)$, where V is a Seifert matrix for the knot, by Theorem 6.2.1. By Theorem 9.4.2, there is an invertible, determinant 1, integer matrix M such that MVM^t if of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$. So the Alexander polynomial is

$$\begin{aligned} \det(V - tV^t) &= \det(M)\det(V - tV^t)\det(M^t) \\ &\quad \text{since } \det(M) = \det(M^t) = 1 \\ &= \det(M(V - tV^t)M^t) \text{ by the Multiplicative} \\ &\quad \text{Property of Determinants} \\ &= \det(MVM^t - tMV^tM^t). \end{aligned}$$

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Corollary 9.4.3 (continued 1)

Proof (continued). Now

$$MVM^t = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \text{ implies } (MVM^t)^t = MV^tM^t = \begin{pmatrix} 0 & C^t \\ B^t & D^t \end{pmatrix}.$$

So the Alexander polynomial is

$$\det(MVM^t - tMV^tM^t) = \det\left(\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} - t\begin{pmatrix} 0 & C^t \\ B^t & D^t \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 0 & B - tC^t \\ C - tB^t & D - tD^t \end{pmatrix}.$$

Now, in general, for partitioned square matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} is square and nonsingular, we have

$$\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})\ldots$$

Corollary 9.4.3 (continued 2)

Proof (continued).

$$\ldots \det(A) = \det(A_{11}) \det(A_{22} - A_{21} A_{11}^{-1} A_{12})$$

(see Theorem 3.4.3 in my online notes for Theory of Matrices [MATH 5090] on Section 3.4. More on Partitioned Square Matrices: The Schur Complement). In fact, we can use a sequence of row interchanges and column interchanges to show that if A_{22} is nonsingular then

$$\det(A) = \det(A_{22}) \det(A_{11} - A_{12} A_{22}^{-1} A_{21})$$

(see Theorem 2.5.C(2) in my online notes for Applied Multivariate Statistical Analysis [STAT 5730] on Section 2.5. Partitioned Matrices).

Corollary 9.4.3 (continued 3)

Corollary 9.4.3. The Alexander polynomial of a slice knot can be factored as $\pm t^k f(t) f(t^{-1})$ for some integer polynomial f and integer k.

Proof (continued). Notice that $D - tD^t$ is nonsingular (or else the Alexander polynomial would be simply 0), so

$$\det(D - tD^t)\det(0 - (B - tC^t)(D - tD^t)^{-1}(C - tB^t))$$
$$= \det(D - tD^t)\det(-(B - tC^t)(D - tD^t)^{-1}(C - tB^t))$$
$$= -\det(D - tD^t)\det(B - tC^t)\det((D - tD^t)^{-1})\det(C - tB^t)$$
$$= -\det(B - tC^t)\det(C - tB^t) \text{ because } \det((D - tD^t)^{-1}) = 1/\det(D - tD^t)$$
$$= -\det(B - tC^t)\det((C - tB^t)^t) = -\det(B - tC^t)\det(C^t - tB)$$
$$= -\det(B - tC^t)\det((-t)(B - (t/t)C^t)$$
$$= \pm t^k \det(B - tC^t)\det(B - (1/t)C^t)$$

for some k. So we can take $f(t) = \det(B - tC^t)$ and the claim holds. □

Corollary 9.4.4

Corollary 9.4.4. If a knot is slice then its signature (and all its ω-signatures) are 0.

Proof. Recall that the signature of a matrix is is the number of positive entries minus the number of negative entries on the diagonal, and for a Seifert matrix V of a knot K, the matrix $V + V^t$ is symmetric and its signature is the signature of knot K, denoted $\sigma(K)$. Livingston only considers the real case, declaring "a proof for the complex signatures [that is, the ω-signatures] is similar."

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By Theorem 9.4.2, there is an invertible, determinant 1, integer matrix M such that MVM^t is of the form $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$. Also,

$$(MVM^t)^t = MV^tM^t = \begin{pmatrix} 0 & C^t \\ B^t & D^t \end{pmatrix}.$$

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Corollary 9.4.4 (continued 1)

Proof (continued). Notice that $M(V+V^t)M^t$ is symmetric (since it equals its transpose), so we have

$$M(V+V^t)M^t = \begin{pmatrix} 0 & S \\ S^t & R \end{pmatrix}$$

for some matrices 0, S, and R.

We borrow a result from another source. The determinant of any knot, $\det(V+V^t)$, is odd and hence is nonzero (see Theorem 3.1 in "Seifert Matrix" on Mitchell Faulk's webpage [accessed 5/2/2021]). So $V+V^t$ is invertible and so $M(V+V^t)M^t = \begin{pmatrix} 0 & S \\ S^t & R \end{pmatrix}$ is invertible.

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Corollary 9.4.4 (continued 2)

Proof (continued). As mentioned in the proof of Corollary 9.4.3, if A_{22} is nonsingular then

$$\det(A) = \det(A_{22})\det(A_{11} - A_{12}A_{22}^{-1}A_{21})$$

(see Theorem 2.5.C(2) in my online notes for Applied Multivariate Statistical Analysis [STAT 5730] on Section 2.5. Partitioned Matrices). So

$$0 \neq \det(V + V^t) = \det(M(V + V^t)M^t) = \det \begin{pmatrix} 0 & S \\ S^t & R \end{pmatrix}$$

$$= \det(R)\det((0) - SR^{-1}S^t) = \pm\det(R)\,\det(S)\det(R^{-1})\det(S^t),$$

and hence $\det(S) \neq 0$, so that matrix S must is invertible (nonsingular).

Corollary 9.4.4 (continued 3)

Proof (continued). So we can use the same row and column operations to put $\begin{pmatrix} 0 & S \\ S^t & R \end{pmatrix}$ into the form $\begin{pmatrix} 0 & I_g \\ I_g & R \end{pmatrix}$; the row operations can be performed by a product of elementary matrices, M, and the the corresponding column operations can be performed by M^t such that $V + V^t = M \begin{pmatrix} 0 & I_g \\ I_g & R \end{pmatrix} M^t$. By Note 6.3.B, the signature of $V + V^t$ and $\begin{pmatrix} 0 & I_g \\ I_g & R \end{pmatrix}$ are the same. Similarly, row and column operations can be used to eliminate the bottom right-hand block to get $\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}$ with the same signature as $V + V^t$ by Note 6.3.B. Now this last matrix has signature 0, and so the slice knot has signature 0, as claimed. □