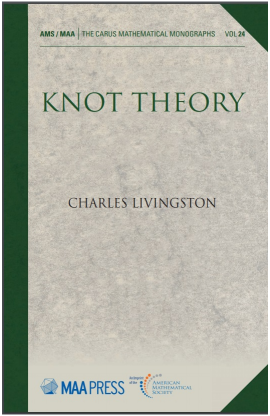
Introduction to Knot Theory

Chapter 9. High-Dimensional Knot Theory

9.5. The Knot Concordance Group—Proofs of Theorems



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Partial Proof. Livingston claims that is K and J are slice then $K \# J$ is slice; he claims this is a "geometric fact" that the "reader should be able to sketch out the argument." To show that $K_1 \# J_1$ is concordant to $K_2 \# J_2$, we need to show $(K_2 \# J_1) \# (K_2 \# J_2)^{rm}$ is slice.

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$$(K_2\#J_1)\#(K_2\#J_2)^{rm} = (K_1\#K_2^{rm})\#(J_1\#J_2^{rm}).$$

This is a connected sum of two slice knots (namely, the slice knots $(K_1\#K_2^{rm})$ and $(J_1\#J_2^{rm})$; these are slice because of the concordance hypothesis). So this knot is slice and hence $K_1\#J_1$ and $K_2\#J_2$ are concordant, as claimed. □

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Proof. The connected sum $\#$ is associative and commutative (as claimed in the proof of Lemma 9.5.6). The identity is the concordance class of the unknot U, since $K\#U = K$ for all knots K. Now the concordance class of the unknot consists of all slice knots, because K and U are concordant if (by definition) $K\#U = K$ is slice.

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