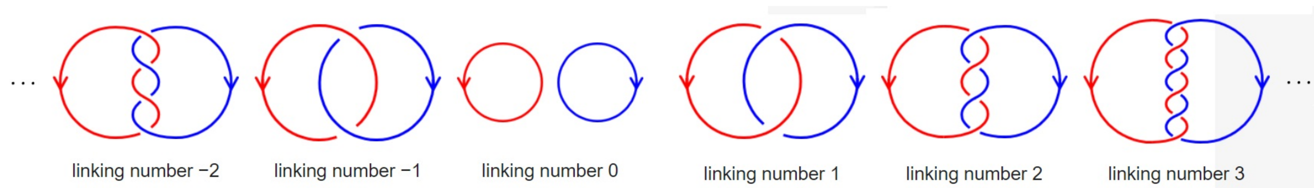
Chapter 1. A Century of Knot Theory

Note. In this section we give some history of knot theory. Some of this information is from the Wikipedia page on "Knot Theory" and the Wikipedia page on "The History of Knot Theory" (accessed 1/27/2021).

Note. A mathematical theory of knots was first discussed by Alexandre-Théophile Vandermonde (February 28, 1735–January 1, 1796) in 1771 in his *Remarques sur des Problèmes de Situation.* In this, he remarked that the topological properties are the important properties in discussing knots (interpret "*situation*" as "position"):

> "Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the manner in which the threads are interlaced." (from Wikipedia page on Vandermonde)

Note. The linking number of two knots is related to the way the knots are intertwined:



From the Wikipedia "Linking Number" Page.

In 1833, Johann Carl Friedrich Gauss (April 20, 1777–February 23, 1855) introduced the Gauss linking integral for computing linking numbers. His student, Johann B. Listing (July 25, 1808–December 24, 1882; Listing was the first to use the term "topology" in 1847), continued the study.

Note. In 1877 Scottish physicist Peter Tait published a series of papers on the enumeration of knots (see P.G. *On Knots*, Scientific Papers, London: Cambridge University Press (1898)). He viewed two knots as "equivalent" if one could be deformed into the other. He wanted to enumerate the non-equivalent knots with a given number of crossings. Appendix 1 gives such an enumeration of graphs with 0 to 9 crossings. Tait, along with Charles H. Little (circa 1858–September 7, 1923) worked on enumerating knots with 10 crossings (there are 165 such knots; see The Rolfson Knot Table on The Knot Atlas webpage).

Note. In this chapter, we think of a knot as a loop of string. In Figure 1.1, we represent knots with a 2-dimensional figure. Since the knot is a 3-dimensional object, we need to reflect depth somehow. We do so at a crossing by drawing the string closer to us as solid (called an "overcrossing" in Section 2.4) and the string farther from us with a gap (called an "undercrossing").

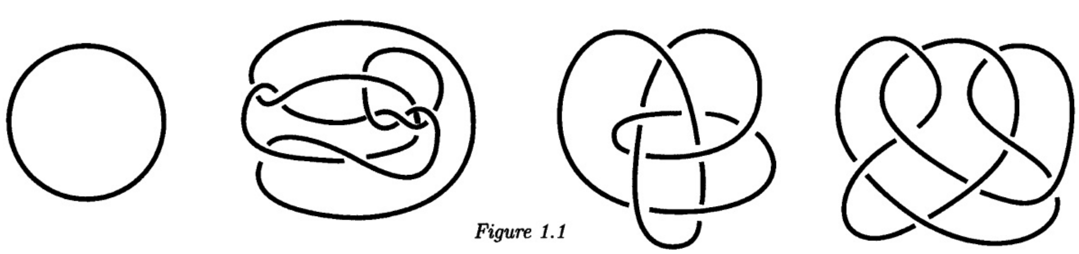


Figure 1.1

The first knot in Figure 1.1 is the the "trivial knot" or the "unknot." The second knot is drawn with 10 crossings, but in fact it is equivalent to the trivial knot which can be drawn with no crossings (notice that this isn't at all clear, and so illustrates the difficulty in determining when two knots are equivalent). The last two knots in Figure 1.1 are not equivalent to the trivial knot (also not obvious). Recognizing when knots are equivalent (the "recognition problem") is fundamental to Tait's enumeration problem. In Section 3.1, "Reidemeister Moves," we introduce ways of manipulating drawings of graphs to modify the crossings and give different drawings of the same ("equivalent") knot.

Note 1.2. With the development of the area of topology around the the turn of the 20th century (largely lead by Henri Poincaré, April 29, 1854–July 17, 1912), the way was open to put knot theory on a rigorous foundation (see my online notes for Introduction to Topology (MATH 4357/5357)). One of the early results was due to Max Dehn (November 13, 1878–June 27, 1952), who proved that the two knots in Figure 1.2 are not equivalent (these are the left- and right-handed trefoil knots, which are "mirror images" of each other; notice that Appendix 1 only contains the right-handed trefoil knot so that it does not include mirror images). His result appeared in Max Dehn, Die beiden Kleeblattschlingen ["The Two Clover Leaf Loops"], *Mathematische Annalen*, **75**(3), 402–413 (1914).

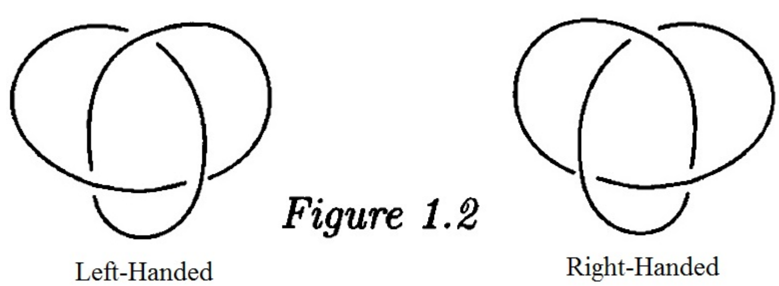
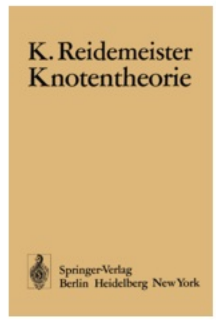


Figure 1.2

Note. In 1928 James W. Alexander (September 19, 1888–September 23, 1971) gave a methods by which a polynomial can be associated with a knot. The results appeared in: J. W. Alexander, Topological Invariants of Knots and Links, *Transactions of the American Mathematical Society*, **30**(2), 275–306 (1928) (available online; accessed 1/27/2021). If one knot is equivalent to another then the knots have the same associated "Alexander polynomial" (it is a "knot invariant"). This is useful in the recognition problem, but there are polynomials which are not equivalent which still have the same Alexander polynomial. In Appendix 2 (which gives all knots with 9 or fewer crossings) there are 8 knots (out of the 87 in the appendix) share polynomials with other knots in the appendix (so the Alexander polynomial is powerful in distinguishing between knots, though not a perfect way to show knots are not equivalent). We explore the Alexander polynomial in Section 3.5.

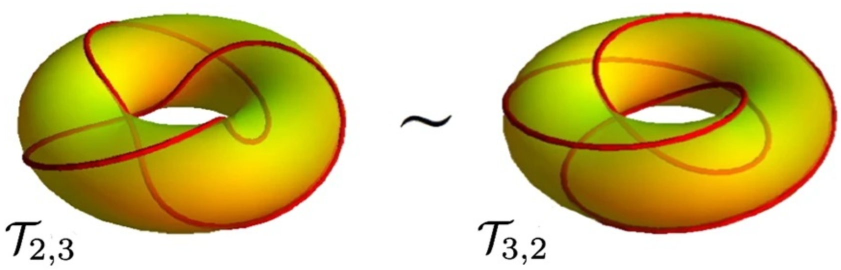
A photo of James Alexander from the
Wikipedia page on Alexander (accessed 1/27/2021).

Note. Kurt W. F. Reidemeister (October 13, 1893–July 8, 1971) published the first book on knot theory, *Knotentheorie*, Ergebnisse der Mathematic, Volume 1, Berlin: Springer-Verlag (1932) (an English translation is available by L. F. Boron, C. O. Christenson, and B. A. Smith, Moscow, ID: BCS Associates (1983)). It is in this book that the "Reidemeister moves" are introduced which allow us to simplify the drawings of a given knot.



Kurt Reidemeister and his book; images from MacTutor History of Mathematics Archive page on Reidemeister and the Springer website.

Note. A "torus knot" is a knot that can be drawn on a torus. For example, the trefoil knot can be drawn on a torus as follows:



From Influence of Winding Number on Vortex Knots Dynamics, *Scientific Reports*, **9**, Article number 17284 (2019).

Notice that the knot goes around the hole in the torus two times and goes through the hole three times, and for this reason the trefoil knot is the $(2, 3)$-torus knot. Figure 1.3 gives a drawing of the $(3, 5)$-torus knot and the $(3, -5)$-torus knot.

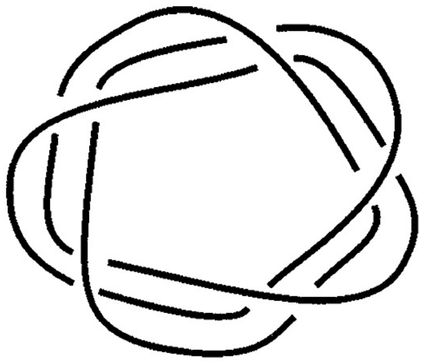
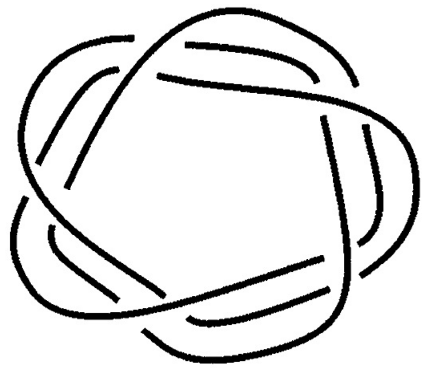
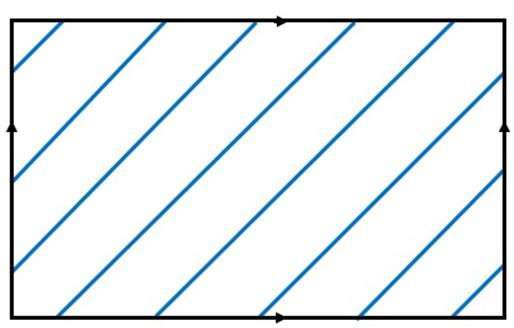
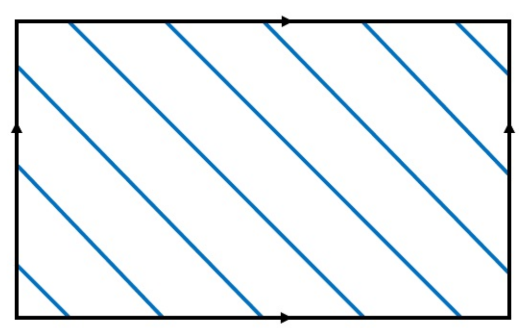


Figure 1.3



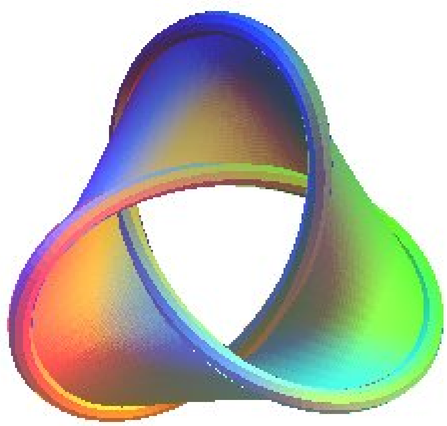
We can also illustrate a torus knot by presenting the "fundamental domain" of the torus and then drawing the knot on this. In the following figure, consider the top and bottom of the rectangles to coincide and the left and right sides of the rectangles to coincide (making the connections as indicated by the arrows). On the left is a representation of the $(3, 5)$-torus knot and on the right is the $(3, -5)$-knot.





For any ordered pair of relatively prime integers, (p, q), with $p > 1$ and $|q| > 1$, there is a corresponding (p, q)-torus knot. M. Dehn and O. Schreier used group theory to prove that the (p, q) and (p', q')-torus knots are the same if and only if the sets $\{p, q\}$ and $\{p', q'\}$ are the same.

Note. In 1934, Herbert Seifert (May 27, 1907–October 1, 1996) showed that if a knot is a boundary of a surface in 3-space then that surface can be used to reflect properties of the knot (he also gave an algorithm to construct a surface bounded by any given knot). For example, the trivial knot determines a circular disk. The Seifert surface for the trefoil knot is in the following figure.



From the MathCurve.com Seifert Surface webpage.

The Seifert surface can be used to compute many of the knot invariants. In addition, it allows the use of geometric and topological methods more directly in the study of knots (versus combinatorial and algebraic methods which had previously been used in such thing as, for example, the Alexander polynomial). Seifert's work appears in: Über das Geschlecht von Knoten ["About the Sex of Knots"], *Mathematische Annalen*, **110**(1), 571–592 (1934).

Note. In 1949, Horst Schubert (June 11, 1919–2001) defined the idea of a prime knot and the decomposition of knots into prime knots. The decomposition is performed with respect to the idea of a "connected sum" of knots. In Figure 1.5, the connected sum of two knots K and J (denoted $K\#J$) is given.

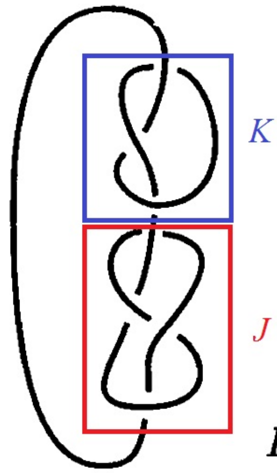


Figure 1.5

So in $K\#J$, we have knot K followed by knot J in the loop of string. A knot is *prime* if it cannot be decomposed as a connected sum of nontrivial knots (Appendix 1 gives the prime knots with 9 or less crossings). H. Schubert proved that any know can be decomposed uniquely as the connected sum of prime knots; from this it follows that if K is nontrivial then there is no knot J so that $J\#K$ is unknotted. His work appeared in: Horst Schubert, Die eindeutige Zerlegbarkeit eines Knotens in Primknoten ["The Unambiguous Decomposability of a Knot into Prime Knots"], *S. Ber. Heidelberger, Akad. Wiss.* **2**, 57–104 (1949).

Note. We can conclude that knots which differ in one invariant are not equivalent. Many initial results in knot theory were of this nature and allowed for showing knots are different; but little was presented which addressed the case when knots are the same. In this direction, Max Dehn in 1910 claimed that (to use Livingston's description of it; see page 6) that "...if a knot were indistinguishable from the trivial knot using algebraic methods, then the knot was in fact trivial." This became known as the Dehn Lemma. The work appeared in: Max Dehn, Über die Topologie des dreidimensionalen Raumes ["About the Topology of Three-Dimensional Space"], *Mathematische Annalen*, **69**, 137–168 (1910). An error in Dehn's proof was found by Hellmuth Kneser (April 16, 1898–August 23, 1973) and pointed out in: Hellmuth Kneser, Gesclossene Flächen in dreidimensionalen Mannigfaltigkeiten ["Closed surfaces in three-dimensional manifolds"], *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **38**, 248–260 (1929). In 1957, Christos D. Papakyriakopoulos (June 29, 1914–June 29, 1976) gave a correct proof of the Dehn Lemma in: Christos Papakyriakopoulos, On Dehn's Lemma and the Asphericity of Knots, *Annals of Mathematics*, **66**, 1-26 (1957) (available through JSTOR—requires you to log in using your ETSU username and password; accessed 1/28/2021). The Dehn Lemma then became the cornerstone of many knot theory results concerning the equivalence of knots. For example, Friedhelm Waldenausen proved in 1968 that two knots are equivalent if and only if "certain algebraic data" associated with the knots are the same. Thid work appears in: Friedhelm Waldenhausen, On Irreducible 3-Manifolds that are Sufficiently Large, *Annals of Mathematics*, **87**, 56–88 (1968) (available through JSTOR—available without login; accessed 1/29/2021).

Note. As an additional example of using graph invariants, it was shown by Kunio Murasugi that if a knot can be drawn so that the crossings alternate from over to under, then the coefficients of the Alexander polynomial alternate in sign. The graph in Figure 1.6 has two consecutive over-crossings (marked with dots). We might wonder if this graph can be redrawn in such a way that crossings alternate.

However, the Alexander polynomial is $2t^6 - 3t^5 + t^4 + t^3 + t^2 - 3t + 2$, which does not have coefficients which alternate in sign. So NO the graph cannot be redrawn with crossings alternating.

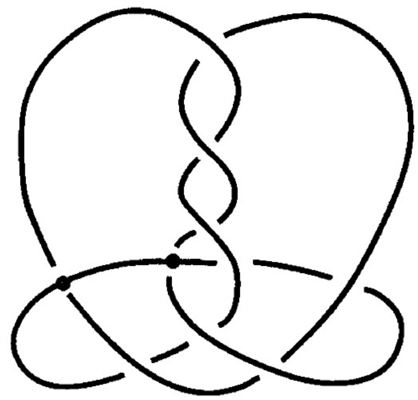
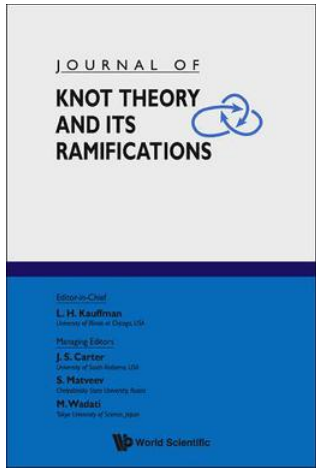


Figure 1.6

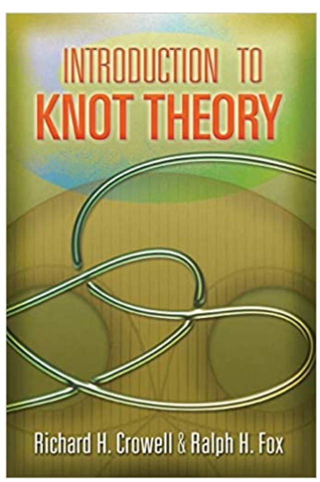
Note. Over the past 50 years, knot theory has grown tremendously. John Horton Conway (December 26, 1937–April 11, 2020; he died in the COVID19 pandemic) introduced combinatorial methods and the Conway polynomial. William Thurston (October 20, 1946–August 21, 2012), a pioneer in the exploration of low dimensional manifolds, has given geometric insights and results. One way to study knots, is to "swell" them up from a 1-dimensional string to a sort-of tube, and then to remove the tube from 3-space yielding the complement of the knot. In 1988, Cameron Gordon and John Luecke proved that knots with equivalent complements are themselves equivalent. Their work appeared in: C. McA. Gordon and J. Luecke, Knots are Determined by their Complements, *Journal of the American Mathematical Society*, **2**, 371–415 (1989) (available from the AMS website; accessed 1/29/2021).

Note. With the blossoming of knot theory in the 1970s and 1980s, the first journal devoted to knot theory, *Journal of Knot Theory and Its Ramifications*, first appeared in March 1992.

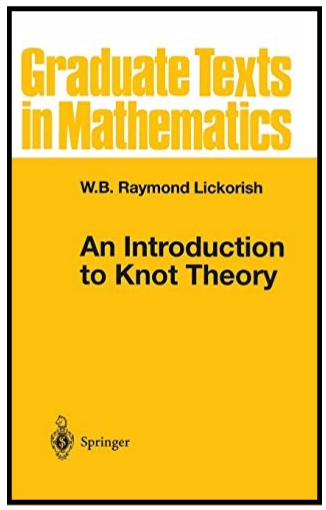


The website for Journal of Knot Theory and Its Ramifications contains information about recent issues (accessed 1/29/2021).

Note. We mentioned the first knot theory text book, Reidemeister's *Knotentheorie* published in 1932. Another foundational book is: Richard Crowell and Ralph Fox, *Introduction to Knot Theory*, NY: Springer Verlag (1963). In fact, this is still in print by Dover Publications.

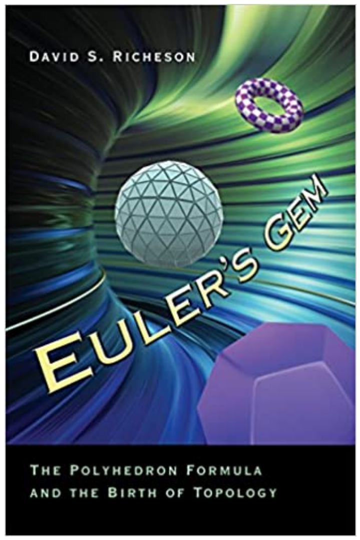


Note. A somewhat more contemporary, graduate-level book is: W. B. Raymond, *An Introduction Knot Theory*, Graduate Texts in Mathematics 175, NY: Springer-Verlag (1997). This gives a rigorous introduction to many of the topics discussed in this chapter and in the Livingston book in general. Topics include: Knot factorization, Jones polynomial, alternating links, Alexander polynomial, covering spaces, Conway polynomial, fundamental group, 3-manifolds, invariants from the Jones polynomial, and Kaufmann polynomials.



Preliminary online notes are available from this source for use in Graduate Knot Theory.

Note. A nice, popular level book that include some knot theory history is David S. Richeson's *Euler's Gem: The Polyhedron Formula and the Birth of Topology*, Princeton, NJ: Princeton University Press (2008); see "Chapter 18. A Knotty Problem." In addition to a history of the "birth" of topology, surfaces (both orientable and nonorientable) are explored and the Classification Theorem for Surfaces is discussed (see "Chapter 16. Rubber Sheets, Hollow Doughnuts, and Crazy Bottles" and "Chapter 17. Are They the Same, or Are They Different?"). This is a "must read" for informal background on topology and knot theory (and a bit of graph theory)!



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