# **Section 10.1.** The Conway Polynomial of a Knot

**Note.** We define the Conway polynomial of a knot (or link) $K$, $\nabla_K(z)$, in terms of the Alexander polynomial. Based on the three types of "crossings" $L_+$, $L_-$, and $L_S$ introduced at the beginning of this chapter, we state a formula (the "skein relation") involving $\nabla_{L_+}$, $\nabla_{L_+}$, and $\nabla_{L_S}$ which we use to find the Conway polynomial for several knots and links.

**Definition.** An Alexander polynomial of a knot $K$ is *normalized* if $A_K(t) = A_K(t^{-1})$ and $A(1) = 1$. A normalized Alexander polynomial can be written in terms of $z = t^{1/2} - t^{-1/2}$ where only positive powers of $z$ appear. We then denote this representation as $\nabla_K(z)$ and it is called the *Conway polynomial* (or the *potential function*) of $K$. This is also sometimes called the *Alexander-Conway polynomial.*

**Example 10.1.A.** Notice that an Alexander polynomial for the trefoil knot is $t^2 - t + 1$ (see Example 3.5.1 and Appendix 2). If we divide this by $t$ then we get $t - 1 + t^{-1}$ and in this polynomial we have $(t - 1 + t^{-1})|_{t=1} = 1$ so this is the normalized version of the Alexander polynomial of the trefoil knot. Notice that with $z = t^{1/2} - t^{-1/2}$, we have $z^2 + 1 = (t^{1/2} - t^{1/2})^2 + 1 = t - 2 + t^{-1} + 1 = t - 1 + t^{1/2}$, so that the Conway polynomial of the trefoil knot is $\nabla_K(z) = z^2 + 1$.

**Example 10.1.B.** In Example 3.5.2, it is shown that an Alexander polynomial of the knot $K = 4_1$ is $t^2 - 3t + 1$. Dividing by $-t$ gives $-t + 3 - t^{-1}$ and in this polynomial $(-t + 3 - t^{-1})|_{t=1} = 1$ so this is the normalized version of the

Alexander polynomial for $K = 4_1$. Notice that with $z = t^{1/2} - t^{-1/2}$, we have $-z^2 + 1 = -(t^{1/2} - t^{-1/2})^2 + 1 = -(t - 2 + t^{-1}) + 1 = -t + 3 - t^{-1}$, so that the Conway polynomial of the knot $K = 4_1$ is $\nabla_K(z) = -z^2 + 1$.

**Note.** In Exercise 10.1.2 it is to be shown that every Alexander polynomial of any knot or link can be normalized. In Corollary 6.2.2 is shown that for knot $K$, normalized Alexander polynomial $A_K(t) = \sum_{i=0}^{n} a_i t^i$ where $a_i = a_{n-i}$, so that the coefficients are symmetric. In Exercise 10.1.A it is to be shown that every such symmetric polynomial of $t$ can written as a polynomial function $z = t^{1/2} - t^{-1/2}$.

**Note.** In Exercise 10.1.2, it is argued that Alexander polynomials for links also have a type of symmetry. The proof is based on Seifert matrices and the genus of the Seifert surface. We get the polynomial in the variable $t^{1/2}$. Similar to Theorem 3.5.6, the Alexander polynomial for a link is not unique but different diagrams produce polynomials that differ by multiples of $\pm(t^{1/2})^k$ for some $k \in \mathbb{N}$.

**Note 10.1.A.** Conway proved that the Conway polynomials of links $L_+$, $L_-$, and $L_S$ (that is, Conway polynomials for diagrams which differ at one crossing with a right-handed crossing for $L_+$, a left-handed crossing for $L_-$, and a crossing that has been smoothed for $L_S$) satisfy the relationship $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$. This is called the "skein relation" (see the Wikipedia Alexander polynomial webpage; accessed 2/5/2021). It was introduced in John H. Conway, "An enumeration of knots and links," in *Computational Problems in Abstract Algebra*, J. Leech (ed.), pp. 329-358, Pergamon (1969).

**Example 10.1.C.** Since the Alexander polynomial of the unknot $U$ is $A_U(t) = 1$, then the Conway polynomial of the unknot is also $\nabla_U(t) = 1$. In Figure 10.2, we have unknots (left and center) with a right-handed crossing (left, $L_+$), a left-handed crossing (center, $L_-$), and the same orientation with a smoothing (right, $L_S$; we denote this "unlink of two components" as $U_2$). So we can use Conway's formula in Note 10.1.A, $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$ to conclude that $-z\nabla_{L_S}(z) = 1 - 1 = 0$ or $\nabla_{L_S}(z) = \nabla_{U_2}(z) = 0$.

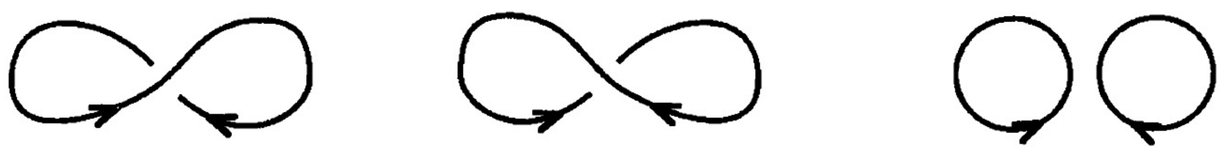

*Figure 10.2*

**Example 10.1.D.** In Figure 10.3, we have the $(2,2)$-torus link, denoted $T_2$, on the left. This is $L_+$ with respect to the uppermost crossing. The center diagram is $L_-$ with respect to the uppermost crossing and is the unlink of two components $U_2$. The right diagram is $L_S$ with respect to the uppermost crossing and is the unknot $U$. By Note 10.1.A, $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$ or $\nabla_{T_2}(z) - \nabla_{U_2}(z) = -z\nabla_U(z)$. Since $\nabla_U(z) = 1$ and $\nabla_{U_2}(z) = 0$ (by the previous Note), then $\nabla_{T_2}(z) = \nabla_{U_2}(z) - z\nabla_U(z) = 0 - z(1) = -z$.

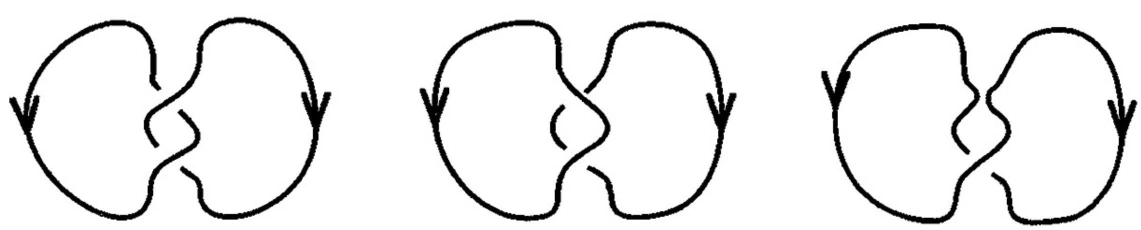

*Figure 10.3*

**Note.** In Example 10.1.A, we computed the Conway polynomial of the trefoil knot $K$ using the Alexander polynomial and we found that $\nabla_K(z) = z^2 + 1$. In Figure 10.4, the trefoil knot is given on the left where the upper crossing is $L_+$, the unknot $U$ is given in the center where the upper crossing is $L_-$, and the $(2, 2)$-torus link $T_2$ where the upper crossing has been smoothed ("$L_S$") on the right. We know $\nabla_U(z) = 1$ and $\nabla_{T_2}(z) = z$, the again using $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$, we have $\nabla_K(z) = \nabla_U(z) - z\nabla_{T_2}(z) = 1 - z(-z) = z^2 + 1$, as before.

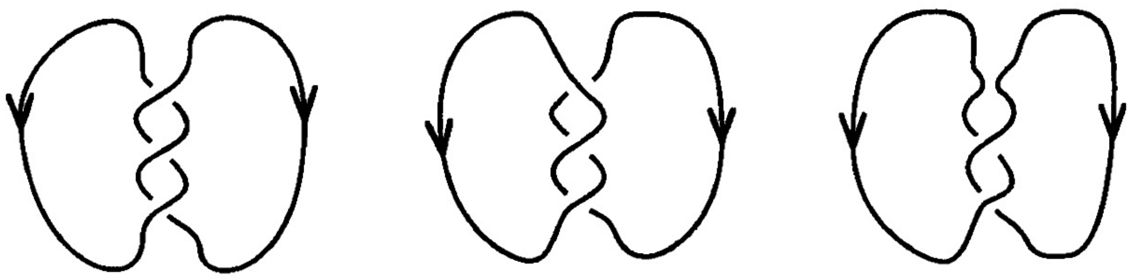

*Figure 10.4*

**Note.** As suggested by the previous examples, we could use Conway's formula $\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$ to build up a catalog of knots and links for which we know the Conway polynomial. This is why Livingston refers the for formula as "recursive." The exercises in this section have you compute the Conway polynomial for several knots and links (see Exercises 10.1.3, 10.1.4, and 10.1.5). The next theorem offers an additional computational approach for Conway polynomials of connected sums, mirror images, reverses of knots.

**Theorem 10.1.1.**

**(a)** For knots $K_1$ and $K_2$, $\nabla_{K_1 \# K_2}(z) = \nabla_{K_1}(z)\nabla_{K_2}(z)$.

**(b)** For any knot $K$, $\nabla_K(z) = \nabla_{K^m}(z) = \nabla_{K^r}(z)$.

**Note.** In Figure 10.5 we have a sequence of knot diagrams.

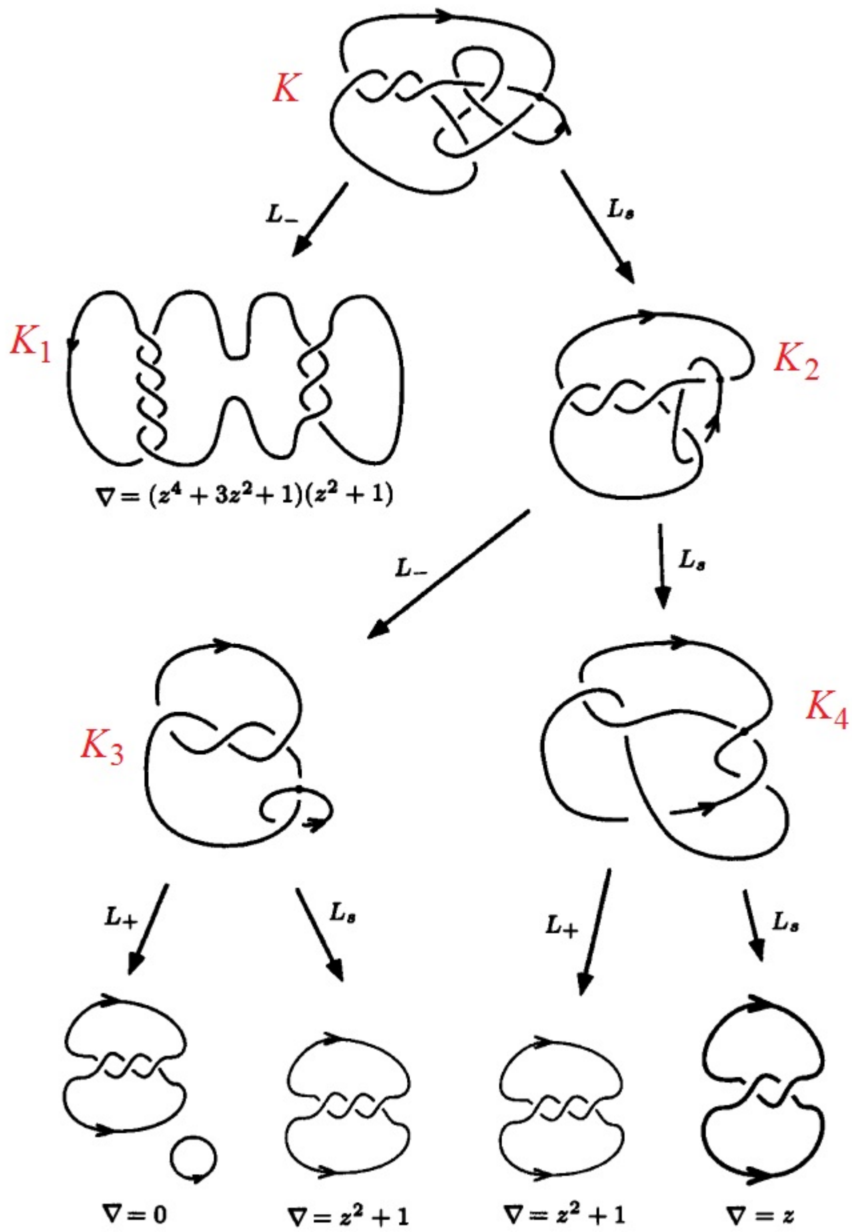


*Figure 10.5*

Notice that the four knots/links at the bottom are (1) a trefoil knot and an unknot for which the Conway polynomial is 0 (we take this as given), (2) the trefoil knot for which the Conway polynomial is $z^2+1$ by Example 10.1.A , (3) the trefoil knot again, (4) the $(2,2)$-torus link $T_2$ with the orientation of one component reversed (so that the Conway polynomial is $-\nabla_{T_2}(z) = -(-z) = z$ by Exercise 10.1.D). Working from the bottom, we have by Conway's formula of Note 10.1.A ($\nabla_{L_+}(z) - \nabla_{L_-}(z) = -z\nabla_{L_S}(z)$) that:

$$(0) - \nabla_{K_3}(z) = -z(z^2+1) \text{ or } \nabla_{K_3}(z) = z^3 + z,$$

$$(z^2+1) - \nabla_{K_4}(z) = -z(z) \text{ or } \nabla_{K_4}(z) = 2z^2+1,$$

$$\nabla_{K_2} - (z^3+z) = -z(2z^2+1) \text{ or } \nabla_{K_2} = -z^3, \text{ and}$$

$$\nabla_K(z) - (z^4+3z^2+1)(z^2+1) = -z(-z^3) \text{ or } \nabla_K(z) = (z^4+3z^2+1)(z^2+1) - z^4$$

$$= (z^6+3z^4+z^2+z^4+3z^2+1) + z^4 = z^6+5z^4+4z^2+1.$$

That is, $\nabla_K(z) = z^6+5z^4+4z^2+1$. Notice that $K_1$ is the connected sum of the $(2,5)$-torus knot and the trefoil knot. By Exercise 10.1.4 it is to be shown that the Conway polynomial of the $(2,5)$-torus knot is $z^4+3z^2+1$ so and by Example 10.1.A the Conway polynomial of the trefoil knot is $z^2+1$, so by Theorem 10.1.1(a) $\nabla_{K_1} = (z^4+3z^2+1)(z^2+1)$. (Notice the typographical error in Livingston's Figure 10.5 where the Conway polynomial of $K_1$ is erroneously given as $(z^4+3z+1)(z^2+1)$; also notice that Livingston's Figure 10.5 has a "$\Delta$" in several places where it should have a "$\nabla$" [the errors are corrected in these notes].)

*Revised: 2/6/2021*